7153_Ceng_M12_C6A_CO_GS_pp330-333.indd 330 posted in 11th pass on 12.07 12/7/11 2:07 PM

Chapter

6

Exponential Functions

LEARNING GOALS

You will be able to develop your algebraic and graphical reasoning skills by

- Describing the characteristics of exponential functions by analyzing their graphs and equations
- Modelling data using exponential functions to solve problems
- Solving problems that involve exponential equations

? Carbon is a naturally abundant element that is found throughout our planet—in the atmosphere, in rocks and soil, in the oceans, and in every living creature. How do archeologists use this element and mathematical principles to determine the age of artifacts?

7153_Ceng_M12_C6A_CO_GS_pp330-333.indd 331 posted in 11th pass on 12.07 12/7/11 2:07 PM

6 Getting Started

Origami

Origami is a form of traditional Japanese art that began in the 1600s. It involves folding a material, such as paper or fabric, to create a three-dimensional object without cutting or gluing, like this praying mantis.

Suppose that you use a sheet of paper to create an origami sculpture. You notice that when you make one fold, you have two layers of paper. You make another fold and count the number of layers. You repeat this process one more time.

How are the number of folds and the number of layers of paper related?

A. Copy and complete the second column of this table.

Number of Folds	Number of Layers Created	
0	1	
1	2	
2		
3		

B. Choose any two numbers under "Number of Layers Created." Determine the product of the two numbers. Write the two numbers and their product as powers of 2.

7153_Ceng_M12_C6A_CO_GS_pp330-333.indd 332 posted in 11th pass on 12.07 12/7/11 2:07 PM

C. Repeat part B two more times with a different pair of numbers.

D. Is there a pattern that relates the exponents of the powers you multiplied to the exponent of the product? Explain.

E. What exponent law does your answer to part D show?

F. In the blank column in the table, summarize your findings using powers of 2. Label the column "Number of Layers as a Power of 2."

G. Choose any two numbers under "Number of Layers Created." Divide the greater number by the lesser number. Write the two numbers and the quotient as powers of 2. Repeat with a different pair of numbers.

H. Examine the exponents of the numbers you wrote in part G. What exponent law do they show?

I. Explain how you can predict the number of layers created given the number of folds made.

J. Suppose that you continued to fold the sheet of paper and count the number of layers you created. Would the pattern you observed continue forever? Explain.

WHAT DO You Think?

Decide whether you agree or disagree with each statement. Explain your decision.

1. When you analyze data, it is useful to rewrite the numbers as powers to see a pattern.
2. When you see a pattern in a data set, you know that the pattern continues forever.
3. During an experiment, Tim measured and recorded the dependent variable. He noticed a rapid increase from one recorded value to the next. Tim claimed that exponents can be used to model his data.

7153_Ceng_M12_C6A_CO_GS_pp330-333.indd 333 posted in 11th pass on 12.07 12/7/11 2:07 PM

6.1 Exploring the Characteristics of Exponential Functions

YOU WILL NEED

- graphing technology

GOAL

Investigate the characteristics of the graphs of exponential functions.

EXPLORE the Math

exponential function

A function of the form

$y = a(b)^x$

where $a \neq 0$, $b > 0$, and $b \neq 1$.

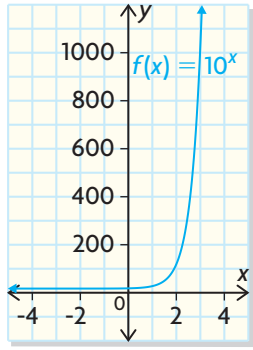
Francine and Kent wondered how the characteristics of **exponential functions** differ from the characteristics of polynomial functions. Francine created a table of values and a graph for each of the functions shown below to investigate similarities and differences.

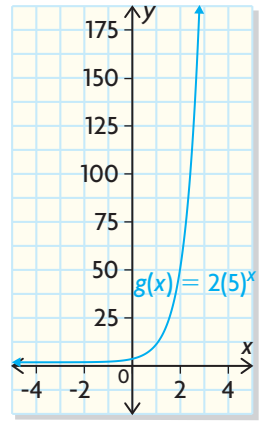
$f(x) = 10^x$

x	$f(x)$
-3	$10^{-3} = 0.001$
-2	$10^{-2} = 0.01$
-1	$10^{-1} = 0.1$
0	$10^0 = 1$
1	$10^1 = 10$
2	$10^2 = 100$
3	$10^3 = 1000$

$g(x) = 2(5)^x$

x	$g(x)$
-3	$2(5)^{-3} = 0.016$
-2	$2(5)^{-2} = 0.08$
-1	$2(5)^{-1} = 0.4$
0	$2(5)^0 = 1$
1	$2(5)^1 = 10$
2	$2(5)^2 = 50$
3	$2(5)^3 = 250$



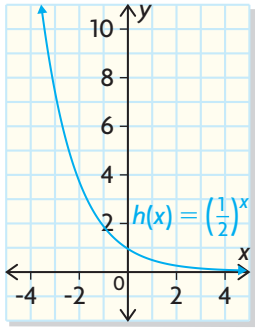


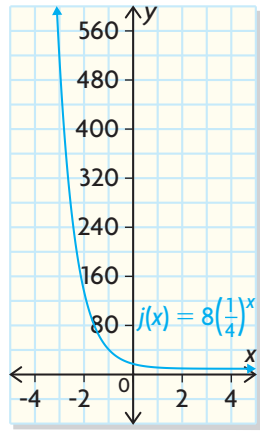
7153_Ceng_M12_C6A_L6.1_pp334-337.indd 334 posted in 12th pass on 12.08 12/8/11 11:01 AM

$h(x) = \left(\frac{1}{2}\right)^x$ $\qquad$ $j(x) = 8\left(\frac{1}{4}\right)^x$

x	$h(x)$
-3	$\left(\frac{1}{2}\right)^{-3} = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = 0.5$
2	$\left(\frac{1}{2}\right)^{2} = 0.25$
3	$\left(\frac{1}{2}\right)^{3} = 0.125$

x	$j(x)$
-3	$8\left(\frac{1}{4}\right)^{-3} = 512$
-2	$8\left(\frac{1}{4}\right)^{-2} = 128$
-1	$8\left(\frac{1}{4}\right)^{-1} = 32$
0	$8\left(\frac{1}{4}\right)^{0} = 8$
1	$8\left(\frac{1}{4}\right)^{1} = 2$
2	$8\left(\frac{1}{4}\right)^{2} = 0.5$
3	$8\left(\frac{1}{4}\right)^{3} = 0.125$





Kent investigated other exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, using his graphing calculator. Both students considered these characteristics of the functions:

- the number of x-intercepts
- the y-intercept
- the end behaviour
- the domain
- the range

? What are the characteristics of the graphs of exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$?

7153_Ceng_M12_C6A_L6.1_pp334-337.indd 335 posted in 11th pass on 12.07 12/7/11 2:06 PM

Reflecting

A. Based on your observations, which characteristics of the graphs of exponential functions are similar to characteristics of polynomial functions you have studied and which are different?

B. Kent claims that all exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, do not have x-intercepts. Do you agree or disagree? Explain.

C. Francine claims that all exponential functions of this form have a restricted range. Do you agree or disagree? Explain.

D. Can you use the end behaviour of exponential functions of this form to help you decide if the function is increasing or decreasing? Explain.

E. Describe any patterns you noticed in the tables of values for the exponential functions you investigated.

In Summary

Key Ideas

- An exponential function has the form $f(x) = a(b)^x$, where x is the exponent and $a \neq 0$, $b > 0$, and $b \neq 1$.
- All exponential functions of the form $f(x) = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, have the following characteristics:

Number of *x*-Intercepts	0
***y*-Intercept**	1
End Behaviour	Curve extends from quadrant II to quadrant I.
Domain	$\{x \mid x \in \mathrm{R}\}$
Range	$\{y \mid y > 0, y \in \mathrm{R}\}$

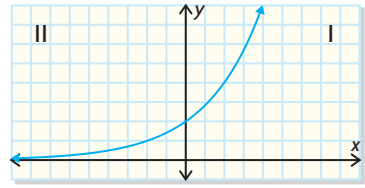
Need to Know

- There are two different shapes of the graphs of an exponential function of the form $f(x) = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$:

  - Case 1: An increasing function; the curve extends from quadrant II to quadrant I.

  - Case 2: A decreasing function; the curve extends from quadrant II to quadrant I.

Increasing



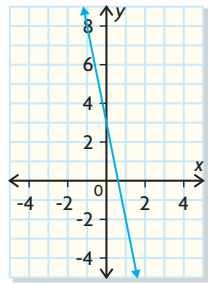
Decreasing

7153_Ceng_M12_C6A_L6.1_pp334-337.indd 336 posted in 12th pass on 12.08 12/8/11 11:01 AM

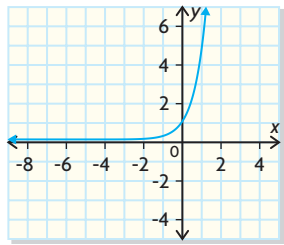
FURTHER Your Understanding

1. Determine whether each graph represents an exponential function. If possible, identify the type of function.

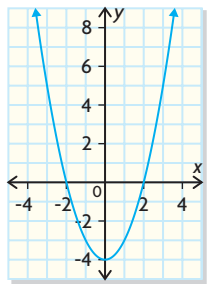
a)



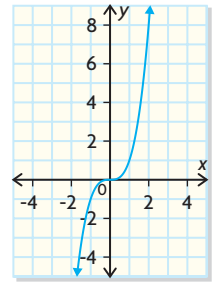
b)



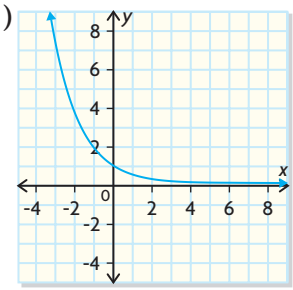
c)



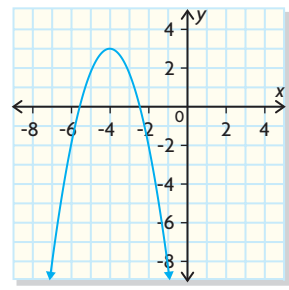
d)



e)



f)



2. For each of the exponential functions you identified in question 1:

- State the number of x-intercepts.
- State the y-intercept.
- State the end behaviour.
- State the domain.
- State the range.

3. Graph each exponential function. Determine the number of x-intercepts, the y-intercept, the end behaviour, the domain, and the range.

a) $y = 10(2)^x$ **b)** $y = 6(2)^x$ **c)** $y = 27\left(\frac{1}{3}\right)^x$ **d)** $y = 4\left(\frac{1}{2}\right)^x$

7153_Ceng_M12_C6A_L6.1_pp334-337.indd 337 posted in 11th pass on 12.07 12/7/11 2:06 PM

6.2 Relating the Characteristics of an Exponential Function to Its Equation

YOU WILL NEED

- graphing technology

EXPLORE...

- Tables of values for three different functions are shown below. How can you use the values to decide which functions are exponential?

x	$f(x)$
1	1
2	4
3	9
4	16
5	25

x	$g(x)$
1	2
2	4
3	8
4	16
5	32

x	$h(x)$
1	1
2	8
3	27
4	64
5	125

It was thought to be impossible to fold a piece of paper, of any size or shape, in half more than eight times. Britney Gallivan proved this incorrect in 2005, when she folded a very long piece of toilet paper 12 times.

GOAL

Predict the characteristics of an exponential function by examining its equation.

INVESTIGATE the Math

Recall the origami activity in Getting Started. Sacha completed a table of values to compare the number of folds, x, to the number of layers of paper created, $f(x)$.

Number of Folds, x	Number of Layers Created, $f(x)$
0	1
1	2
2	4
3	8
4	16
5	32
6	64

He determined that the function

$$f(x) = 1(2)^x$$

can be used to predict the number of layers that will be created for a given number of folds.

? How can you predict the characteristics of an exponential function of the form $f(x) = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, by looking at its equation?

A. Graph Sacha's function:

$$f(x) = 1(2)^x$$

B. Examine the values of $f(x)$ in a table of values, where the x-values increase by 1 each time. How are the values of $f(x)$ related? Which parameter in the equation is associated with this relationship?

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 338 posted in 12th pass on 12.08 12/8/11 11:16 AM

C. Examine the graph and table of values of your function. Determine the y-intercept, and state whether the graph is increasing or decreasing as you move from left to right along the x-axis. Record your results in a table like the one below.

Characteristics	$f(x) = 1(2)^x$
y-intercept	
Increasing or decreasing?	

D. Choose a new exponential function where $a > 0$, $b > 0$, and $b \neq 1$. Graph the new function, and repeat parts B and C. Add this information to your table.

E. Repeat part D for several more exponential functions where $a > 0$, $b > 1$, and $b \neq 1$.

F. Consider an exponential function where $a > 0$ and $0 < b < 1$, such as

$$f(x) = 2\left(\frac{1}{3}\right)^x$$

Repeat parts B and C. Add this information to your table.

G. Repeat part F for several more exponential functions where $a > 0$ and $0 < b < 1$.

H. Examine the table you have created and the parameters a and b in each function.

i) Can you predict the y-intercept from the parameters in the function? Explain.

ii) Can you predict whether the graph of the function will increase or decrease? Explain.

Reflecting

I. Consider a table of values for an exponential function, where values of the independent variable increase by 1. What relationship exists between consecutive values of the dependent variable? Why does this relationship occur?

J. Suppose that you are given the equation of an exponential function of the form

$$y = a(b)^x$$

where $a > 0$, $b > 0$, and $b \neq 1$.
Explain how you can determine:

i) the y-intercept

ii) whether the function is increasing or decreasing

K. Consider the general exponential function given in part J. Does changing either of the parameters a or b affect the end behaviour, domain, or range? Explain.

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 339 posted in 11th pass on 12.07 12/7/11 2:05 PM

APPLY the Math

EXAMPLE 1 Connecting the characteristics of an increasing exponential function to its equation and graph

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, and the range of the following function:

$$y = e^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing of the function.

Communication *Tip*

The symbol e is a constant known as Euler's number. It is an irrational number that equals 2.718.... This number occurs naturally in some situations where a quantity increases continuously, such as increasing populations.

Sharon's Solution

The function

$$y = e^x$$

can also be written as

$$y = 1(2.718...)^x$$

This function is an exponential function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$.

In this function, $a = 1$ and $b = e$ or 2.718....

I determined that this is an exponential function, since the variable x is an exponent. I also determined the values of parameters a and b.

Common Characteristics

Number of x-intercepts: 0

End behaviour: The curve extends from quadrant II to quadrant I.

Domain: $\{x \mid x \in \mathrm{R}\}$

Range: $\{y \mid y > 0, y \in \mathrm{R}\}$

I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for exponential functions of this form. These functions are defined for all real numbers, so this gave me the domain. I also knew that these functions have a restricted range bounded by the x-axis. This means the graph only occurs in quadrants II and I and has no x-intercepts.

Unique Characteristics

$y = 1(2.718...)^0$

$y = 1(1)$

$y = 1$

y-intercept: 1

I determined the y-intercept by substituting $x = 0$ into the function and determining y.

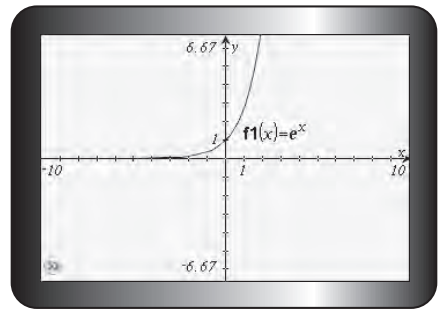
I noticed that the parameter a in the equation is also equal to 1. I predict that a is always the y-intercept of the function, since any base to the exponent 0 is always 1.

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 340 posted in 12th pass on 12.08 12/8/11 11:16 AM

The function increases as you move from left to right along the x-axis.

I predicted that the function increases, since parameter b in the equation is greater than 1.

I can use a graphing calculator to create a table of values and a graph to verify my predictions.



My predictions were correct.

Your Turn

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, and the range of this function:

$$f(x) = 2(5)^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing the function.

EXAMPLE 2 Connecting the characteristics of a decreasing exponential function to its equation and graph

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, the range, and whether this function is increasing or decreasing:

$$y = 9\left(\frac{2}{3}\right)^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing the function.

Pat's Solution

The function $y = 9\left(\frac{2}{3}\right)^x$ is an exponential function of the form $y = a(b)^x$, where $a > 0$ and $0 < b < 1$.

I determined that this is an exponential function, since the variable x is an exponent. I also determined the values of parameters a and b.

In this function, $a = 9$ and $b = \frac{2}{3}$.

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 341 posted in 11th pass on 12.07 12/7/11 2:05 PM

Common Characteristics

Number of x-intercepts: 0

End behaviour: The curve extends from quadrant II to quadrant I.

Domain: $\{x \mid x \in \mathrm{R}\}$

Range: $\{y \mid y > 0, y \in \mathrm{R}\}$

I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$.

Unique Characteristics

$$y = 9\left(\frac{2}{3}\right)^0$$

$$y = 9(1)$$

$$y = 9$$

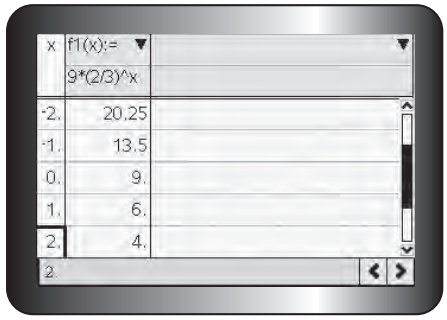
y-intercept: 9

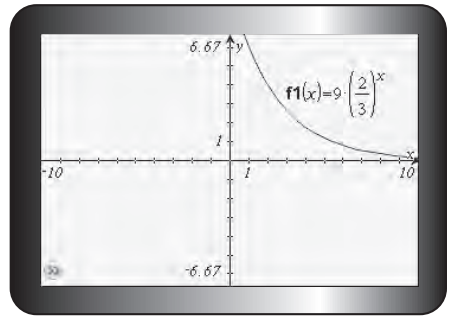
I determined the y-intercept by substituting $x = 0$ into the function and determining y. The parameter a in the equation has the same value as the y-intercept.

The function decreases as you move from left to right along the x-axis.

I predicted that the function decreases, since parameter b in the equation is greater than 0 and less than 1.

I can use a graphing calculator to create a table of values and a graph to verify my predictions.





My predictions were correct.

Your Turn

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, the range, and whether this function is increasing or decreasing:

$$f(x) = 8\left(\frac{3}{4}\right)^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing the function.

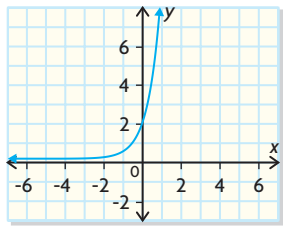
7153_Ceng_M12_C6A_L6.2_pp338-351.indd 342 posted in 12th pass on 12.08 12/8/11 11:18 AM

EXAMPLE 3 Matching an exponential equation with its corresponding graph

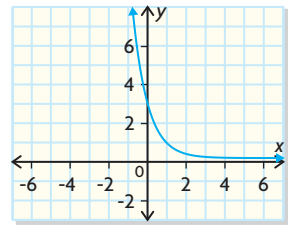
Which exponential function matches each graph below? Provide your reasoning.

i) $y = 3(0.2)^x$ **ii)** $y = 4(3)^x$ **iii)** $y = 4(0.5)^x$ **iv)** $y = 2(4)^x$

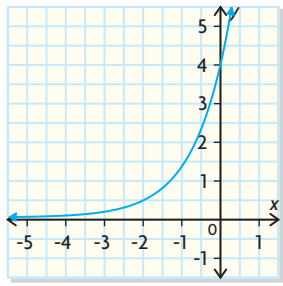
a)



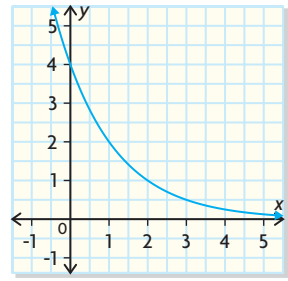
c)



b)

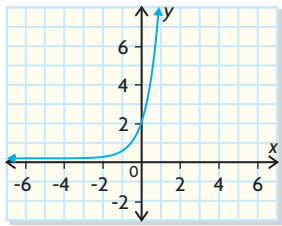


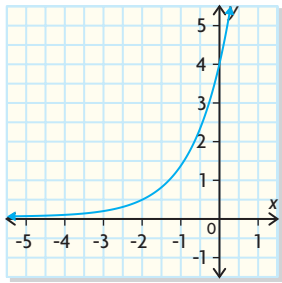
d)



Arthur's Solution

Graphs a) and b) are of increasing functions. Since they are increasing, the functions for these graphs must have $b > 1$.



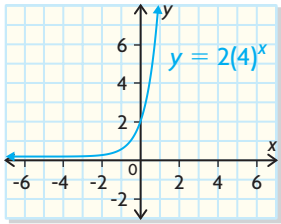


When parameter b is greater than 1, I knew that the graph of a function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, increases as you move from left to right along the x-axis.

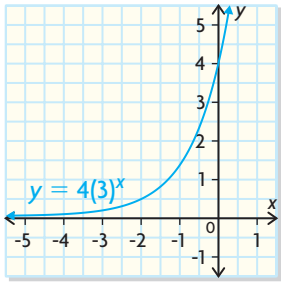
7153_Ceng_M12_C6A_L6.2_pp338-351.indd 343 posted in 12th pass on 12.08 12/8/11 11:18 AM

There are two possible functions for these two graphs:
$y = 4(3)^x$ and $y = 2(4)^x$
y-intercept: 4 y-intercept: 2

I knew that parameter a, in a function of the form $y = a(b)^x$, represents the y-intercept of the function. So in the function $y = 4(3)^x$, the y-intercept is 4, and in the function $y = 2(4)^x$, the y-intercept is 2. I can use the y-intercepts of the graphs of the two increasing functions to match the graphs to the equations.

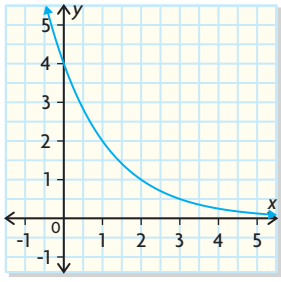


This graph passes through 2 on the y-axis, so its function must be $y = 2(4)^x$.

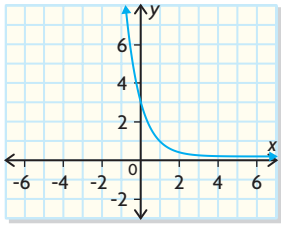


This graph passes through 4 on the y-axis, so its function must be $y = 4(3)^x$.

The remaining two graphs are of decreasing functions.
Since they are decreasing, the functions for these graphs must have $0 < b < 1$.



When parameter b is greater than 0 and less than 1, I knew that the graph of a function of the form $y = a(b)^x$, where $a > 0$, decreases as you move from left to right along the x-axis.



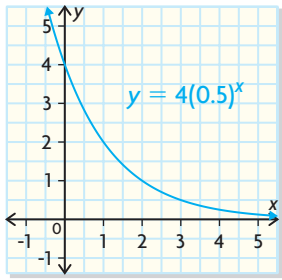
7153_Ceng_M12_C6A_L6.2_pp338-351.indd 344 posted in 12th pass on 12.08 12/8/11 11:18 AM

There are two possible equations for these two graphs:

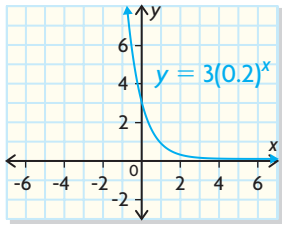
$y = 3(0.2)^x$ and $y = 4(0.5)^x$

y-intercept: 3 y-intercept: 4

I knew that parameter a, in a function of the form $y = a(b)^x$, represents the y-intercept of the function. So in the function $y = 3(0.2)^x$, the y-intercept is 3, and in the function $y = 4(0.5)^x$, the y-intercept is 4. I used the y-intercepts of the graphs of the two decreasing functions to match the graphs to the equations.



This graph passes through 4 on the y-axis, so its function must be $y = 4(0.5)^x$.



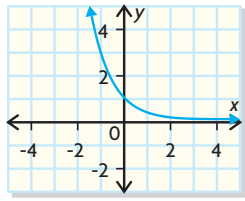
This graph passes through 3 on the y-axis, so its function must be $y = 3(0.2)^x$.

Your Turn

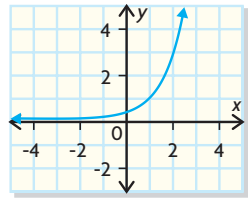
Match each function with the corresponding graph below. Provide your reasoning.

i) $y = (3)^x$ **ii)** $y = \frac{1}{3}(3)^x$ **iii)** $y = 3\left(\frac{1}{3}\right)^x$ **iv)** $y = \left(\frac{1}{3}\right)^x$

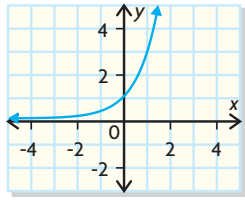
a)



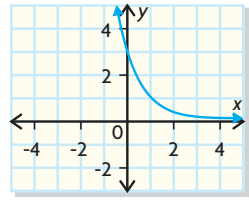
c)



b)



d)



7153_Ceng_M12_C6A_L6.2_pp338-351.indd 345 posted in 11th pass on 12.07 12/7/11 2:05 PM

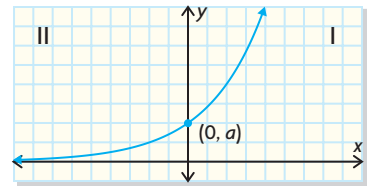
In Summary

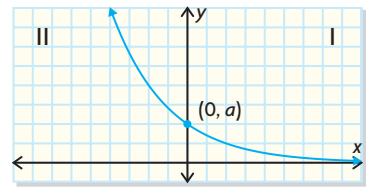
Key Ideas

- In a table of values for an exponential function, there is a constant ratio between consecutive y-values when the x-values increase by the same amount. The value of this ratio is equal to the parameter b in the function $y = a(b)^x$, where $b \neq 1$.
- In an exponential function of the form $y = a(b)^x$, a is a non-zero multiplier and b is the base (where $b > 0$ and $b \neq 1$). The value of a is the y-intercept of the graph of the function.

Need to Know

- An exponential function is an increasing function if $a > 0$ and $b > 1$.
- An exponential function is a decreasing function if $a > 0$ and $0 < b < 1$.
- Changing the parameters a and b in exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, does not change the number of x-intercepts, the end behaviour, the domain, or the range of the function. These characteristics are identical in all exponential functions of this form.





CHECK Your Understanding

1. Determine if the data in each table represents an exponential function. Provide your reasoning.

a)

x	0	1	2	3	4	5
y	7	14	28	56	112	224

b)

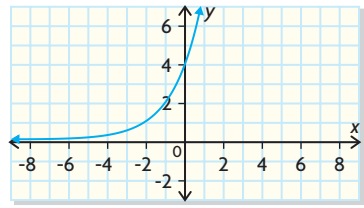
x	0	1	2	3	4	5
y	3072	768	192	48	12	3

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 346 posted in 12th pass on 12.08 12/8/11 11:18 AM

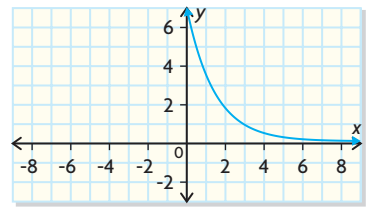
2. Describe the characteristics of the following exponential graphs by completing a table like the one below.

Number of *x*-Intercepts	*y*-Intercept	End Behaviour	Domain	Range

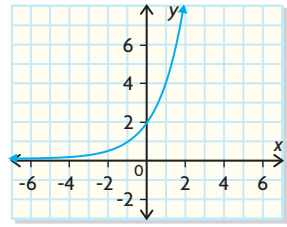
a)



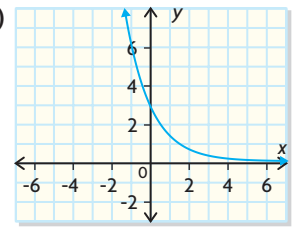
c)



b)



d)



3. What is the difference between an increasing exponential function and a decreasing exponential function? Sketch a graph of each type of function to illustrate your answer.

4. Determine the *y*-intercept of each exponential function, and state whether the function is increasing or decreasing.

a) $y = 5(2)^x$ **b)** $y = 2\left(\frac{1}{2}\right)^x$ **c)** $y = 10(1.5)^x$ **d)** $y = (0.4)^x$

PRACTISING

5. For each set of data:

i) Determine if an exponential function could be used to model the data, and explain how you know;

ii) Identify the *y*-intercept, and state if the function is increasing or decreasing or both.

a)

x	*y*
0	1
1	2
2	4
3	8
4	16
5	32

b)

x	*y*
−4	−5
−3	−3
−2	−1
−1	1
0	3
1	5

c)

x	*y*
−1	256
0	64
1	16
2	4
3	1
4	0.25

d)

x	*y*
−2	5
−1	2
0	1
1	2
2	5
3	−10

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 347 posted in 11th pass on 12.07 12/7/11 2:05 PM

6. Predict the characteristics of each function, based on the given equation. Record your predictions in a table like the one below. Then verify your predictions by graphing each function.

Number of *x*-Intercepts	*y*-Intercept	End Behaviour	Domain	Range

a) $y = 3(5)^x$

b) $y = 4(0.3)^x$

c) $y = 2\left(\frac{1}{5}\right)^x$

d) $y = 3.5\left(\frac{1}{8}\right)^x$

e) $y = 25(1.7)^x$

f) $y = 12(0.8)^x$

7. Explain how you could use the *y*-intercept and base of each function to determine if the function increases or decreases.

a) $y = \frac{1}{3}(8)^x$ **b)** $y = 4\left(\frac{3}{5}\right)^x$ **c)** $y = 0.5(e)^x$

8. Predict the characteristics of each exponential function. Record your predictions in a table like the one below.

Number of *x*-Intercepts	*y*-Intercept	End Behaviour	Domain	Range

a) $y = 4(2)^x$

b) $y = 8\left(\frac{1}{4}\right)^x$

c) $y = 3(3)^x$

d) $y = 10\left(\frac{1}{2}\right)^x$

e) $y = 30(1.8)^x$

f) $y = (0.5)^x$

g) $y = 3(e)^x$

h) $y = 45\left(\frac{2}{5}\right)^x$

9. Determine if any of these are decreasing exponential functions without graphing them. Explain how you know.

a) $f(x) = 7\left(\frac{2}{3}\right)^x$

b) $g(x) = 0.6(2.5)^x$

c) $h(x) = 0.4(0.5)^x$

d) $k(x) = 5(2)^x$

10. Predict which functions are increasing and which are decreasing, without graphing them. Verify your predictions by graphing the functions.

a) $y = 5(2)^x$

b) $y = 20\left(\frac{1}{4}\right)^x$

c) $y = 6(8)^x$

d) $y = 12\left(\frac{1}{3}\right)^x$

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 348 posted in 12th pass on 12.08 12/8/11 11:18 AM

11. Examine each function to determine if it increases or decreases. Then match each function with the corresponding graph below.

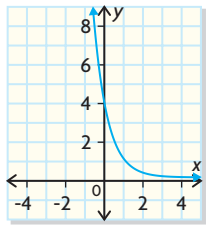
a) $y = 6(1.5)^x$

b) $y = 4(0.25)^x$

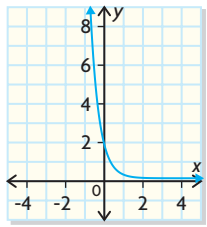
c) $y = 2(0.1)^x$

d) $y = 3(1.15)^x$

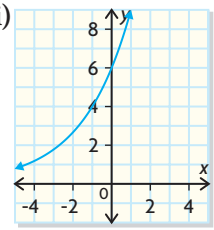
i)



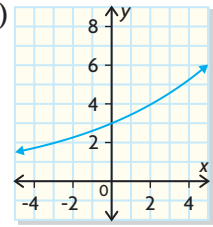
ii)



iii)



iv)



12. Copy and complete this table. Verify your answers by graphing.

	Function	*y*-Intercept	Base	Domain	Range	Increasing or Decreasing
a)	$y = 9(7)^x$					
b)	$y = 7(4)^x$					
c)	$y = 6\left(\frac{1}{7}\right)^x$					
d)	$y = 2(0.35)^x$					
e)	$y = 2(e)^x$					

13. a) Determine the range of each function, and state the values of a and b. Use this information to decide if the function increases or decreases.

i) $y = 2(0.5)^x$

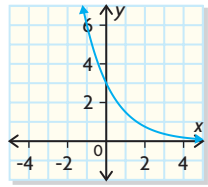
ii) $y = 1(3)^x$

iii) $y = 3\left(\frac{1}{2}\right)^x$

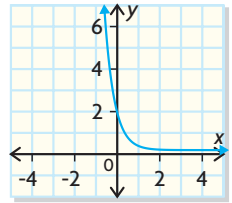
iv) $y = 2(4)^x$

b) Match each function in part a) with the corresponding graph below. Provide your reasoning.

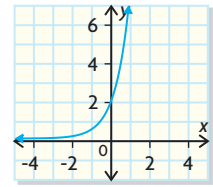
A.



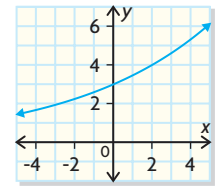
B.



C.



D.



7153_Ceng_M12_C6A_L6.2_pp338-351.indd 349 posted in 11th pass on 12.07 12/7/11 2:05 PM

14. **a)** Write the equations of an increasing exponential function and a decreasing exponential function that each have a y-intercept of 5.

b) Which characteristics of the functions would be the same? Which characteristics would be different?

15. Natasha claims that if she is given the equation of a polynomial function and the equation of an exponential function she can identify which is which. Do you agree or disagree? Justify your decision.

16. Linear functions are defined by $f(x) = ax + b$, where $a \in \mathrm{R}$ and $b \in \mathrm{R}$. The exponential functions you have studied are defined by $f(x) = a(b)^x$, where $a > 0$, $b > 0$, $b \neq 1$, and $a \in \mathrm{R}$ and $b \in \mathrm{R}$. Compare the number of x-intercepts, y-intercept, end behaviour, domain, range, increasing or decreasing for both types of functions. Create a list of characteristics that are similar and a list of characteristics that are different.

Closing

17. You are given an exponential function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, and are asked to identify the number of x-intercepts, the y-intercept, end behaviour, domain, range, and whether the function increases or decreases. Which of these characteristics are unique to the function you have been given, and which of these characteristics are common to all exponential functions of this form?

Extending

18. Two students consumed energy drinks that contained caffeine. The mass of caffeine, y, in milligrams, in each student's body was then measured over an 8 h period. The data is shown in the graphs at left, where x represents time in hours.

a) Use each graph to determine the y-intercept, domain, and range of the function.

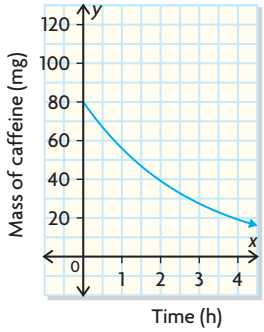
b) Why are both functions decreasing?

c) Based on the graphs, which student consumed more caffeine? Which student's body processed the caffeine more quickly?

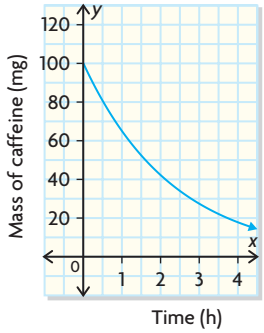
d) Estimate the amount of caffeine in each student's body 4 h after consuming the energy drink.

e) Explain the meaning of the y-intercept in this context. Why are the values of the y-intercept different?

Student A



Student B



7153_Ceng_M12_C6A_L6.2_pp338-351.indd 350 posted in 12th pass on 12.08 12/8/11 11:18 AM

19. Three student council members booked professional hockey player René Bourque to make a presentation at their school. By the end of the first hour after they made the booking, each student had told the news to two of their friends. By the end of the second hour, each of their friends had told the news to two more friends. The pattern continued for several hours.

a) Create a table of values to model the spread of the news during the first 5 h.

b) Can an exponential function be used to model the data? Explain.

c) Use your table of values to define a function that models the data. What is the domain and range of your function?

d) Graph your function. From your graph, extrapolate the number of people who would have heard the news if the pattern had continued through 8 h.

René Bourque, a Franco-Albertan of Métis heritage, was acquired by the Calgary Flames of the NHL in 2008.

20. Philippe Sly was nominated for the music award Jeune soliste 2012. When the news of his nomination broke, bloggers, newspapers, and television stations wanted to interview him. By 9:00 a.m. on Monday morning, Philippe had received four requests for an interview. The number of interview requests he received then doubled every 3 h.

a) Kendra claims that an exponential function can be used to model this situation. Is she correct? Explain.

b) Define an exponential function of the form $y = a(b)^x$ to model Philippe's situation. State what a, b, x, and y represent in this context.

c) What is the domain and range of the function in this context?

d) Extend the following table to determine how many interview requests Philippe had received by 4:00 p.m. on Monday afternoon. Explain your process.

Time	*x*	*y*
9:00 a.m.		
10:00 a.m.		

Philippe Sly, of Ottawa, Ontario, is a member of the Canadian Opera Company Ensemble Studio.

7153_Ceng_M12_C6A_L6.2_pp338-351.indd 351 posted in 11th pass on 12.07 12/7/11 2:05 PM

6.3 Solving Exponential Equations

YOU WILL NEED

- graphing calculator

EXPLORE...

- Is there a way to predict which values of x will give integer values of y in an exponential function?
  a) For each function below, create a table of values that contains only integer values of y.
     i) $y = 5(2)^{\frac{x}{1}}$
     ii) $y = 5(2)^{\frac{x}{3}}$
  b) For each function, how are the x-values that give integer values of y related to the exponent?
  c) Predict which x-values would give integer values of y for the following function:
     $y = 5(2)^{\frac{x}{5}}$

GOAL

Determine the solution to an exponential equation using a variety of strategies.

LEARN about the Math

A chessboard has 64 squares, which alternate between two different colours. There is an interesting fable associated with the invention of the game of chess. According to the fable, the king was so pleased by this game that he let the inventor choose his payment. The inventor asked for grains of rice on each square of the board, according to this pattern: 1 grain of rice on the first square, 2 grains of rice on the second square, 4 on the third square, and so on, until all 64 squares were covered. This pattern can be modelled by the exponential function

$$y = a(b)^{x-1}$$

where x represents the number of the square and y represents the number of grains of rice on that square.

64	63	62	61	60	59	58	57
49	50	51	52	53	54	55	56
48	47	46	45	44	43	42	41
33	34	35	36	37	38	39	40
32	31	30	29	28	27	26	25
17	18	19	20	21	22	23	24
16	15	14	13	12	11	10	9
1	2	3	4	5	6	7	8

? On which square of the chessboard will there be 128 grains of rice?

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 352 posted in 11th pass on 12.07 12/7/11 2:04 PM

EXAMPLE 1 Determining and solving an exponential equation

Determine the square on the chessboard that would correspond to 128 grains of rice.

Hussain's Solution: Solving by interpolating from a graph

$y = a(b)^{x-1}$

$y = a(2)^{x-1}$

Each consecutive square on the board has double the grains of rice on the preceding square. I knew that this pattern involves an increasing exponential function, so $b > 1$. Since this involves a doubling pattern, I knew that the base in the exponential function is $b = 2$.

$1 = a(2)^{1-1}$

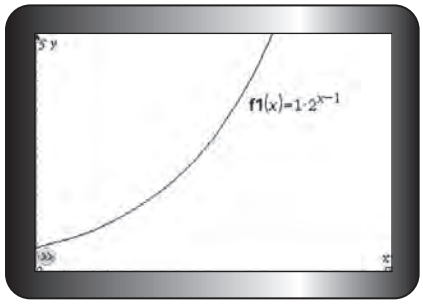
$1 = a(2)^{0}$

$1 = a(1)$

$1 = a$

I also knew that the first square corresponds to $x = 1$ and must have one grain of rice, so $y = 1$. I substituted these values into my function and solved for a.

$y = 1(2)^{x-1}$ is the function that models the number of grains of rice, y, on each square, x.

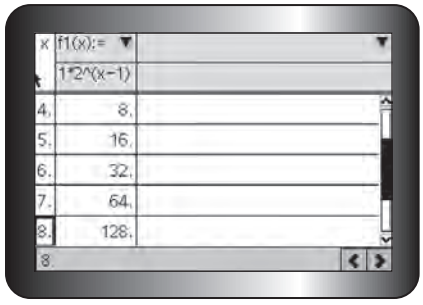


I graphed the function.

Since x represents the number of the square on the chessboard, the domain is the set of natural numbers, $\{x \mid x \in \mathrm{N}\}$.

Since y represents the number of grains of rice, the range is also the set of natural numbers, $\{y \mid y \in \mathrm{N}\}$.

This meant that I only needed to view the graph in quadrant I for positive x-values and y-values.



Since exponential functions, where $b > 1$, grow very quickly, I decided to read data points from the table, rather than trying to find an appropriate viewing window for the graph. I scrolled down the list of x-values to locate the point that has 128 as its y-coordinate.

$x = 8$

There will be 128 grains of rice on the 8th square of the chessboard.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 353 posted in 11th pass on 12.07 12/7/11 2:04 PM

Elliot's Solution: Solving by rewriting each side using the same base

$y = a(b)^{x-1}$
$y = 1(2)^{x-1}$

I knew that the grains of rice on the chessboard involve an increasing exponential function with a base of 2. I also knew that the first square has one grain of rice, so $a = 1$ when $x = 1$ and $y = 1$.

$128 = (2)^{x-1}$

I had to determine the number of the square on the chessboard that has 128 grains of rice. So, I substituted $y = 128$ into my exponential function. This resulted in an exponential equation.

Write 128 as a power of 2.
$128 = (2)(2)(2)(2)(2)(2)(2)$
$128 = 2^7$

I knew that I could solve the equation if I could express both sides of the equation as powers with the same base, in this case, powers of 2.

Substitute 2^7 for 128.
$2^7 = (2)^{x-1}$
$7 = x - 1$
$x = 8$

Since the bases on both sides of the equation are the same, their exponents must also be equal for both sides of the equation to be equal.

There will be 128 grains of rice on the 8th square of the chessboard.

Raessa's Solution: Solving a system of equations using graphing technology

$y = a(b)^{x-1}$
$y = 1(2)^{x-1}$

I knew that the pattern of grains of rice on the chessboard involve an increasing exponential function with a base of 2. I also knew that the first square has one grain of rice, so $a = 1$, when $x = 1$ and $y = 1$.

$128 = (2)^{x-1}$

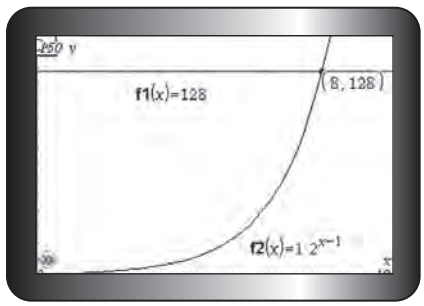
I had to determine the number of the square on the chessboard that had 128 grains of rice. So, I substituted 128 for y in my function. This gave me an exponential equation.

$y_1 = 128$
$y_2 = 1(2)^{x-1}$

I decided to solve the equation by using a system of equations strategy, graphing each side as a function.

The function created using the left side is linear. The function created using the right side is exponential. I entered the equations into my calculator and solved the system graphically. The x-coordinate of the point of intersection is the solution to the exponential equation $128 = (2)^{x-1}$.

The point of intersection is (8, 128), so the solution to my equation is 8.



$x = 8$

There will be 128 grains of rice on the 8th square of the chessboard.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 354 posted in 11th pass on 12.07 12/7/11 2:04 PM

Reflecting

A. Which of the three methods shown do you prefer for solving an exponential equation? Explain why. For this method, verify that the left side of the equation is equal to the right side.

B. Do you think all three methods could be used in every situation? Explain.

C. Given that a typical 1 kg bag of rice contains about 64 000 grains, on which square of the chessboard would there be the equivalent of 1 kg of rice? Does your answer match an actual square on the chessboard? Explain.

APPLY the Math

EXAMPLE 2 Solving exponential equations when bases are powers of one another

Solve each equation, and verify your answer by substitution.

a) $2(4)^{2x} = \frac{1}{32}$ **b)** $5^{x+2} = \sqrt{125}$ **c)** $9^{x-1} = 3^{x-3}$

Denise's Solution

a)

$$2(4)^{2x} = \frac{1}{32}$$

I noticed that the left side of the equation is a product of 2 and $(4)^{2x}$. So, to isolate $(4)^{2x}$ on the left side, I knew that I needed to divide both sides of the equation by 2. I knew that dividing by 2 is the same as multiplying by $\frac{1}{2}$, so I decided to multiply both sides of the equation by $\frac{1}{2}$.

$$\frac{1}{2} \times 2(4)^{2x} = \frac{1}{2} \times \frac{1}{32}$$

$$4^{2x} = \frac{1}{64}$$

Since the left side of the equation is expressed as a power of 4, I wrote 64 as a power of 4.

$$4^{2x} = \frac{1}{4^3}$$

I rewrote the fraction on the right side using a negative exponent. This resulted in an equation with powers of 4 on both sides. The two sides of the equation will be equal only when the exponents are equal.

$$4^{2x} = 4^{-3}$$

$$2x = -3$$

$$x = -\frac{3}{2}$$

$$x = -1.5$$

I solved for x.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 355 posted in 12th pass on 12.08 12/8/11 11:31 AM

Verify $x = -1.5$:

LS	RS
$2(4)^{2x}$	$\frac{1}{32}$
$2((4)^{2(-1.5)})$	
$2(4)^{-3}$	
$2\left(\frac{1}{64}\right)$	
$\frac{2}{64}$	
$\frac{1}{32}$	

LS = RS

I verified my answer by substituting $x = -1.5$ into the given equation and evaluating the left and right sides separately.

b) $5^{x+2} = \sqrt{125}$

$5^{x+2} = \sqrt{5^3}$

I noticed that the left side of the equation has a base of 5. So, I expressed 125 as a power of 5.

$5^{x+2} = (5^3)^{\frac{1}{2}}$

I knew that the exponential form of a square root is a power involving an exponent of $\frac{1}{2}$.

$5^{x+2} = (5)^{3\times\frac{1}{2}}$

I used the power of a power law to simplify the right side of the equation.

$5^{x+2} = 5^{\frac{3}{2}}$

$x + 2 = \frac{3}{2}$

Since the bases on both sides of the equation are both 5, the exponents must also be equal.

$x = \frac{3}{2} - 2$

$x = \frac{3}{2} - \frac{4}{2}$

$x = -\frac{1}{2}$

I solved for x.

I rewrote the right side of the equation using a common denominator of 2.

Verify $x = -\frac{1}{2}$:

LS	RS
5^{x+2}	$\sqrt{125}$
$(5)^{\left(-\frac{1}{2}+2\right)}$	$(125)^{\frac{1}{2}}$
$(5)^{\frac{3}{2}}$	$(5^3)^{\frac{1}{2}}$
	$(5)^{\frac{3}{2}}$

LS = RS

I verified my answer by substituting $x = -\frac{1}{2}$ into the given equation and evaluating the left and right sides separately.

On the left side, I simplified the exponent by adding $-\frac{1}{2}$ and 2 to get the rational exponent $\frac{3}{2}$.

I rewrote the right side of the equation in exponential form.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 356 posted in 12th pass on 12.08 12/8/11 11:31 AM

c) $9^{x-1} = 3^{x-3}$

$(3^2)^{x-1} = 3^{x-3}$

I noticed that the right side of the equation has a base of 3, and I knew that 9 could be expressed as a power of 3. So, I expressed the left side of the equation as a power of 3.

$3^{2(x-1)} = 3^{x-3}$

$3^{2x-2} = 3^{x-3}$

I applied the power of a power law to the left side of the equation. I did this in two steps to avoid making a distributive error.

$2x - 2 = x - 3$

Since the bases on both sides of the equation are both 3, the exponents must also be equal.

$2x - x = -3 + 2$

$x = -1$

I solved for x by collecting like terms.

Verify $x = -1$:

LS	RS
9^{x-1}	3^{x-3}
9^{-1-1}	3^{-1-3}
9^{-2}	3^{-4}
$\frac{1}{9^2}$	$\frac{1}{3^4}$
$\frac{1}{81}$	$\frac{1}{81}$

LS = RS

I verified my answer by substituting $x = -1$ into the given equation and evaluating the left and right sides separately.

Your Turn

a) Solve the equation $4^{x+5} = 64^{2x}$, and verify your answer by substitution.

b) In what situation might you encounter problems if you were solving exponential equations using the method shown in Example 2?

EXAMPLE 3 Determining the half-life of an exponential function

When diving underwater, the light decreases as the depth of the diver increases. On a sunny day off the coast of Vancouver Island, a diving team recorded 100% visibility at the surface but only 25% visibility 10 m below the surface. The team determined that the visibility for the dive could be modelled by the following **half-life exponential function**:

$$A(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$$

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 357 posted in 11th pass on 12.07 12/7/11 3:37 PM

A_0 represents the percentage of light at the surface of the water; x represents the depth in metres; h represents the depth, in metres, where there is half the original visibility; and $A(x)$ represents the percentage of light at a depth of x metres. At what depth will the visibility be half the visibility at the surface?

half-life exponential function

A function of the form

$$y = a\left(\frac{1}{2}\right)^{\frac{x}{h}}$$

where the base is $\frac{1}{2}$, $a \neq 0$, and $h \neq 0$; the value of h is called the half-life because it corresponds to the point on the graph of the function where the value of the function is half its initial value, a.

Jason's Solution: Solving an exponential equation by using the same base

$$A(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$$

$$25 = 100\left(\frac{1}{2}\right)^{\frac{10}{h}}$$

I knew that $A_0 = 100$ and $A(x) = 25$ when $x = 10$. I substituted these values into the half-life function and this resulted in an exponential equation.

$$\frac{25}{100} = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

To solve for h, I divided both sides by 100.

$$\frac{1}{4} = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

I simplified the fraction on the left side to get 1 in the numerator.

$$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

Since the right side of the equation has a base of $\frac{1}{2}$, and $\frac{1}{4}$ can be written with a base of $\frac{1}{2}$, I expressed the left side of the equation as a power of $\frac{1}{2}$.

$$2 = \frac{10}{h}$$

Since the bases on both sides of the equation are both $\frac{1}{2}$, the exponents must also be equal.

$$2h = 10$$
$$h = 5$$

I solved for h.

The visibility at a depth of 5 m will be half the visibility at the surface of the water.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 358 posted in 12th pass on 12.08 12/8/11 11:44 AM

Alyssa's Solution: Solving an exponential equation graphically

$$A(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$$

$$25 = 100\left(\frac{1}{2}\right)^{\frac{10}{h}}$$

From the context, I knew that $A_0 = 100$ and $A(x) = 25$ when $x = 10$. I substituted these values into the function.

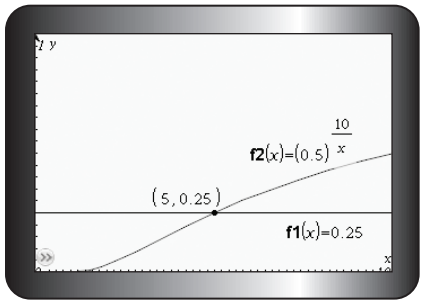
$$0.25 = \left(\frac{1}{2}\right)^{\frac{10}{h}}$$

I divided both sides of the equation by 100.

$$y_1 = 0.25$$

$$y_2 = (0.5)^{\frac{10}{h}}$$

To solve the resulting exponential equation, I used a system of equations strategy. I used my graphing calculator to create the linear function y_1 and the exponential function y_2. I then graphed both functions on the same axes to locate the point of intersection. I knew that the x-coordinate of the point of intersection was the solution to my exponential equation.



$$x = 5$$

The point of intersection is (5, 0.25), so the solution to my equation is 5.

The visibility at a depth of 5 m will be half the visibility at the surface of the water.

Your Turn

On another day, the same diving team noticed that the diving conditions were much better. They recorded 100% visibility at the surface and 3.125% visibility 8 m below the surface.

a) At what depth will the visibility now be half the visibility at the surface?

b) Explain why you chose the method you used in part a).

EXAMPLE 4 Solving exponential equations when the bases are not powers of one another

Solve the following exponential equation:

$$2^{x+1} = 5^{x-1}$$

Round your answer to one decimal place.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 359 posted in 12th pass on 12.08 12/8/11 11:44 AM

Carey's Solution

$2^{x+1} = 5^{x-1}$

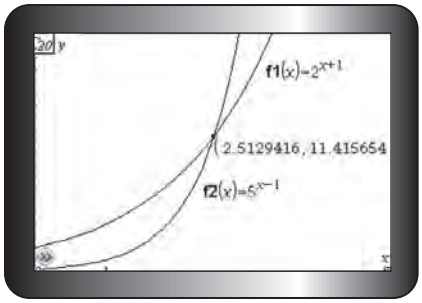
The bases of the powers on each side of the equation are different. In this case, I can't write either base as a power of the other, so the algebraic strategy of using like bases won't work. The only strategy I can use is the system of equations strategy.

$y_1 = 2^{(x+1)}$ $\quad y_2 = 5^{(x-1)}$

To solve this exponential equation I used my graphing calculator to create the exponential function y_1 and another exponential function y_2.

$2^{(2+1)} = 8$ $\quad 5^{(2-1)} = 5$
$2^{(3+1)} = 16$ $\quad 5^{(3-1)} = 25$

I substituted 2 and 3 into each function. This gave me a range of y-values for both functions that overlap. This means the point of intersection must occur for a value of x between 2 and 3.



I graphed both functions on the same axes to locate the point of intersection. I knew that the x–coordinate of the point of intersection was the solution to my exponential equation.

The solution is $x = 2.5$, to one decimal place.

The point of intersection is (2.512..., 11.415...), so the solution to my equation is 2.5.

Your Turn

a) Solve this exponential equation: $3^{x-5} = 2(5)^{\frac{x}{4}}$
Round your answer to one decimal place.

b) Describe one convenience of using graphing technology to solve this equation. Describe one difficulty you encountered.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 360 posted in 11th pass on 12.07 12/7/11 2:05 PM

In Summary

Key Ideas

- Two exponential expressions with the same base are equal if their exponents are equal. For example, if $a^m = a^n$, then $m = n$, where $a > 0$, $a \neq 1$, and $m, n \in \mathrm{R}$.
- An exponential equation can be solved algebraically or graphically.

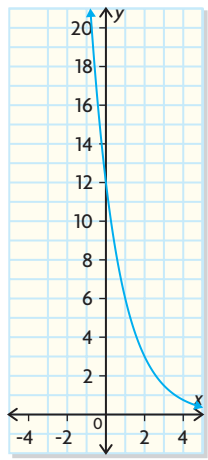
Need to Know

- To estimate the solution to an exponential equation, you can enter the exponential equation as a system of equations on a graphing calculator and graph the system. You can then determine the point of intersection. The x–coordinate of the point of intersection is the solution to the exponential equation.
- To determine an exact solution to an exponential equation algebraically, write both sides of the equation as powers with the same base (if possible). Then set the exponents equal to each other, and solve the resulting equation.
- Sometimes it is not possible to write both sides of an exponential equation as powers using the same base, or it is not obvious how to do so. In these situations you can solve an exponential equation using the systems of equations strategy.

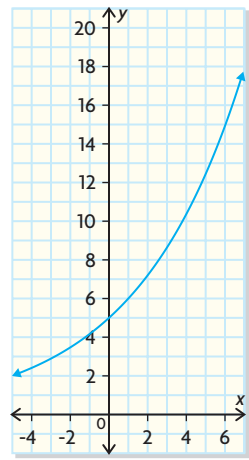
CHECK Your Understanding

1. For each exponential function shown, estimate the x-values at $y = 3$ and $y = 15$.

a)



b)



2. Solve each exponential equation by expressing both sides with the same base. Verify your answer by substitution.

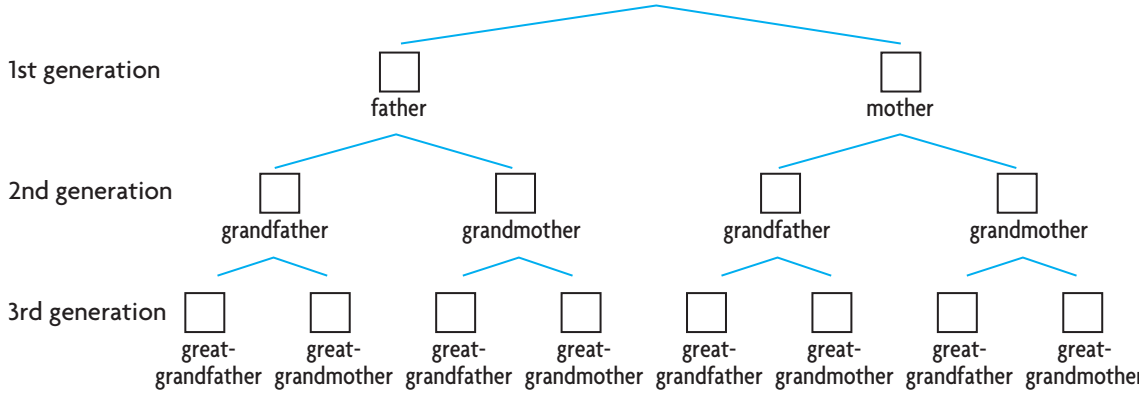
a) $8^x = 64$ **b)** $2^t = 1$ **c)** $13^n = \frac{1}{169}$ **d)** $3^x = \sqrt{27}$

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 361 posted in 12th pass on 12.08 12/8/11 2:48 PM

3. Joshua has been examining his family tree. His two parents are his ancestors from one generation before, his four grandparents are his ancestors from two generations before, and his eight great-grandparents are his ancestors from three generations before. To model the number of ancestors, he used the following equation:

$$y = 1(2)^x$$

a) Is this an increasing or decreasing function? Explain how you know.
b) What do x and y represent in this context? State the domain and range.
c) Assuming the same pattern, how many ancestors did Joshua have seven generations ago?
d) Assuming the same pattern, in which generation were there 4096 ancestors?



PRACTISING

4. Solve each equation by expressing both sides with the same base.

a) $10^x = 1000$
b) $2^{-t} = 32$
c) $3^n = \frac{1}{81}$
d) $3^{3x+1} = 81$
e) $2^{x+2} = (8)^{\frac{1}{2}}$
f) $4^{3x} = \frac{1}{16}$

5. Solve each exponential equation by using graphing technology.

a) $5^{2x} = \left(\frac{1}{5}\right)^{x+2}$
b) $9^{1-x} = 27^{x+2}$
c) $\left(\frac{1}{16}\right)^{2x} = 4^{x-5}$

6. **a)** Solve the following equation by choosing a common base of 8, 4, or 2 for both sides.

$$8^{3x-2} = 64^{x+1}$$

b) Choose a different base to solve the equation. Did you get the same answer?

7. Solve each equation by expressing both sides as powers with the same base.

a) $3^{x+1} = 9$
b) $8(3)^{\frac{x}{2}} = 72$
c) $5^{3x+2} = 5$
d) $\sqrt{8} = 2^{4x+1}$
e) $4^{1-2x} = 32$
f) $10^{2(x-3)} = 1000$

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 362 posted in 11th pass on 12.07 12/7/11 2:05 PM

8. Solve each equation by using graphing technology. Round your answers to one decimal place.

a) $3(3^x) = \frac{1}{4}$

b) $4(2^x) = 42$

c) $2^{x+5} = 6^x$

d) $7^{x-1} = \left(\frac{1}{2}\right)(2)^{x+2}$

9. Ryan is working with two exponential functions. The first is an increasing function:

$$y = 3^x$$

The second is a decreasing function:

$$y = \left(\frac{1}{2}\right)^{x-4}$$

a) Explain how Ryan can determine a point that is common to both functions.

b) Determine the coordinates of the common point. Round to the nearest tenth.

10. The radioactive element cobalt-60 can be used to treat cancer patients. Radioactive elements decay into other elements in a predictable way over time. The percent of cobalt-60, $A(t)$, left in a sample can be modelled by the half-life function

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.3}}$$

where t represents the time, in years, after the initial time and A_0 represents the initial amount, 100%, of the cobalt-60.

a) How long does it take for a sample of cobalt-60 to reduce to half its initial amount?

b) What percent of cobalt-60 will remain in a sample after 10 years?

c) Determine how long it will take, to the nearest tenth of a year, until only 10% of the cobalt-60 remains.

11. The population of a specific bacteria growing in a Petri dish is modelled by the function

$$P(t) = 12\,000(3)^{\frac{t}{4}}$$

where $P(t)$ represents the number of bacteria and t represents the time in hours after the initial count.

a) Determine the value of t when $P(t) = 324\,000$ by rewriting the equation with the same base.

b) What does your answer for part a) mean in this context? Explain.

c) Determine the value of t when $P(t) = 14\,000$ using graphing technology. Round your answer to two decimal places.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 363 posted in 11th pass on 12.07 12/7/11 2:05 PM

12. The length of time that a particular type of blue cheese will keep in a refrigerator can be modelled by the exponential function

$$H(T) = 130\left(\frac{1}{2}\right)^{\frac{T}{43.3}}$$

where $H(T)$ represents the number of hours after the cheese is put in the refrigerator and T represents the temperature of the refrigerator in degrees Celsius.

a) If the refrigerator is set at a temperature of 8 °C, how many hours will the cheese keep? Round your answer to the nearest tenth.

b) If you want the cheese to keep for 5 days, at what temperature should the refrigerator be set? Round your answer to the nearest tenth of a degree.

13. Lazy Den manufactures and sells its own furniture. To clear out old stock, Lazy Den is having a sale. The price of all existing stock will be reduced by 20% at the start of each week until all the stock is sold. The exponential function that models the price of a chair from the start of the sale is

$$C(t) = 2500(0.8)^t$$

where $C(t)$ represents the cost of the chair at the end of the week and t represents the time in weeks since the sale began.

a) What was the initial price of the chair? Explain how you know.

b) After which week did the chair cost less than half of its initial price? Explain how you arrived at your answer.

c) Eric bought the chair for $655.36. In which week did Eric buy the chair?

14. A vehicle was purchased for $24 000 in 2002. The book value of the vehicle can be modelled by the exponential function

$$V(t) = 24\,000(0.85)^t$$

where $V(t)$ represents the book value in dollars and t represents time in years since the purchase date.

a) How long after the purchase date will the vehicle be worth $12 000?

b) Is it possible to determine the year that your answer for part a) corresponds to? Explain.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 364 posted in 12th pass on 12.08 12/8/11 11:55 AM

15. A biologist is studying the growth of a particular type of bacteria in a Petri dish. At noon, there are 300 bacteria in the dish. The population doubles every 12 h. The population of the bacteria, $P(h)$, at any hour, h, can be modelled by the following function:

$$P(h) = 300(2)^{\frac{h}{12}}$$

How long will it take for this bacterial population to grow to 307 200?

16. Living organisms contain traces of carbon-14, which is radioactive. While an organism is alive, the amount of carbon-14 stays constant. When the organism dies, however, the percent of the original carbon-14 left in its body can be modelled by the following exponential function:

$$A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

Since carbon-14 has a half-life of about 5730 years, archeologists can use the percentage, $A(t)$, of the remaining carbon-14 to determine the approximate age of fossils, bones, and artifacts made from living things, such as leather or wooden tools. Determine the age, to the nearest year, of an artifact that has

a) 75% carbon-14 remaining

b) 25% carbon-14 remaining

c) 10% carbon-14 remaining

Closing

17. How is graphing systems of equations related to solving exponential equations algebraically? Justify your answer with an example.

Extending

18. In Example 2 of this lesson, Denise solved $2(4)^{2x} = \frac{1}{32}$ by first dividing both sides of the equation by 2. Solve the equation by multiplying both sides by 64.

7153_Ceng_M12_C6A_L6.3_pp352-365.indd 365 posted in 12th pass on 12.08 12/8/11 11:55 AM

6 Mid-Chapter Review

FREQUENTLY ASKED Questions

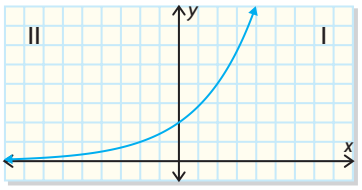
Study *Aid*
- See Lessons 6.1 and 6.2.
- Try Mid-Chapter Review Questions 1 and 6.

Q: How can you recognize when a graph, a table of values, or an equation represents an exponential function?

A: **Graph:** If you are given a graph, you can tell if the function is exponential by the shape of the graph. All the exponential functions you have studied are one of these two curves:

- an increasing curve that extends from quadrant II to quadrant I

Increasing



- a decreasing curve that extends from quadrant II to quadrant I

Decreasing

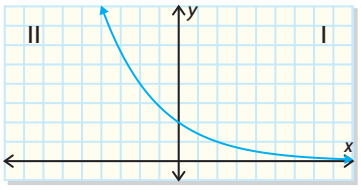


Table of values: If you are given a table of values in which all the x-values increase by the same amount, calculate the ratios for consecutive pairs of y-values. If the ratios are the same, then the values represent an exponential function.

x	y	
0	5	
1	15	$\frac{15}{5} = 3$
2	45	$\frac{45}{15} = 3$
3	135	$\frac{135}{45} = 3$
4	405	$\frac{405}{135} = 3$
5	1215	$\frac{1215}{405} = 3$

In this table of values, the y-values increase by a factor of 3, so the values represent an exponential function.

Equation of a function: If you are given an equation, look at the form of the equation. All the exponential functions you have studied are of the form $y = a(b)^x$, where the independent variable, x, is the exponent in the equation.

7153_Ceng_M12_C6A_MCR_pp366-369.indd 366 posted in 12th pass on 12.08 12/8/11 12:02 PM

Q: How are the characteristics of exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, different from the characteristics of polynomial functions of degree 3 or less?

A: **Number of x-intercepts:** Exponential functions of this form have no x-intercepts. Polynomial functions of degree 3 or less may have 1, 2, or 3 x-intercepts.

End behaviour: Exponential functions of this form extend from quadrant II to quadrant I. Polynomial functions of degree 3 or less can extend from quadrant III to quadrant I, quadrant II to quadrant IV, quadrant II to quadrant I, or quadrant III to quadrant IV.

Range: Exponential functions of this form have a restricted range: $\{y \mid y > 0, y \in \mathrm{R}\}$. Only polynomial functions of degree 2 have a restricted range.

Study **Aid**

- See Lesson 6.2, Examples 1 and 2.
- Try Mid-Chapter Review Questions 3 to 5.

Q: How can you solve an exponential equation?

A: If the equation models the context of a problem, then you can graph the function and either interpolate or extrapolate the appropriate values.

If the equation can be written so that both sides are powers with like bases, then the equation can be solved algebraically. For example:

$2^{x+1} = 64$

$2^{x+1} = 2^6$ Write 64 as a power of 2.

$x + 1 = 6$ Since the bases on both sides of the equation are equal, the exponents are also equal.

$x = 6 - 1$ Solve for x.

$x = 5$

If the equation cannot be written so that both sides are powers with like bases, then you can set up a system of equations on a graphing calculator and determine the point of intersection. The x-coordinate of the point of intersection is the solution to the exponential equation. For example:

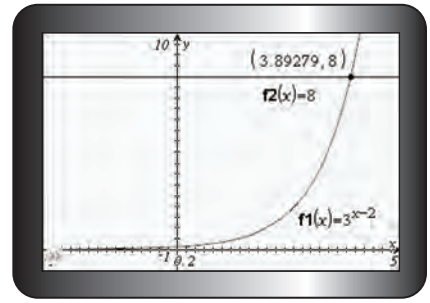
$3^{x-2} = 8$ 8 cannot be written as a power of 3.

$y_1 = 3^{x-2}$ Create a system of equations, and enter the system into a graphing calculator.

$y_2 = 8$

Graph the system, and determine the x-coordinate of the point of intersection.

$x = 3.892...$



Study **Aid**

- See Lesson 6.3, Examples 1 to 4.
- Try Mid-Chapter Review Questions 7 to 11.

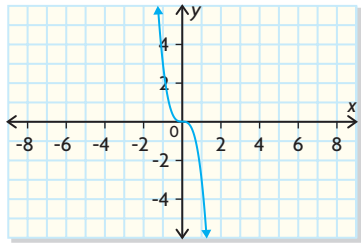
7153_Ceng_M12_C6A_MCR_pp366-369.indd 367 posted in 11th pass on 12.07 12/7/11 2:02 PM

PRACTISING

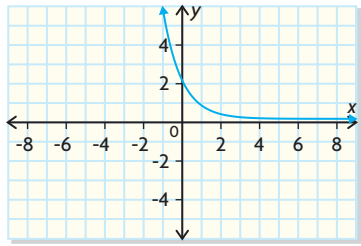
Lesson 6.1

1. Determine whether each graph represents an exponential function. Explain how you know.

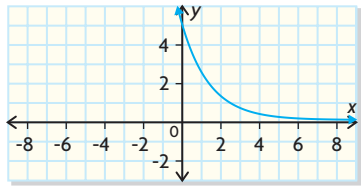
a)



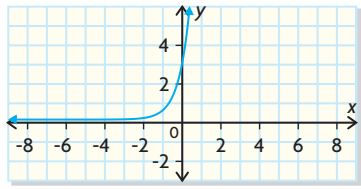
b)



c)



d)



2. a) Graph each function using technology.

i) $y = 3(4)^x$ **iii)** $y = 5(3)^x$

ii) $y = 8\left(\frac{1}{2}\right)^x$ **iv)** $y = 2\left(\frac{1}{8}\right)^x$

b) List the characteristics of each function, including the number of x-intercepts, the y-intercept, end behaviour, domain, and range.

Lesson 6.2

3. State the number of x-intercepts, the y-intercept, end behaviour, domain, and range for each function, without graphing the function. Predict whether the function is increasing or decreasing. Verify your answers by graphing.

a) $y = 5(2)^x$ **b)** $y = 90\left(\frac{1}{3}\right)^x$ **c)** $y = 8e^x$

4. For each exponential function in question 1, state the following:

a) the value of a

b) whether the value of b is greater than 1 or between 0 and 1

c) the domain and range

5. Without graphing, predict which exponential functions are increasing and which are decreasing. Verify your predictions by graphing.

a) $y = 4\left(\frac{1}{3}\right)^x$ **c)** $y = 9\left(\frac{1}{4}\right)^x$

b) $y = 3(5)^x$ **d)** $y = 0.3(2)^x$

6. Lauren is president of the student council. She wants every student in the school to vote either for or against the proposed student dress code. Lauren has designed a handout that will be distributed to the seven council members. Each of the council members will then distribute it to seven of their friends who have not already been polled, and so on, until all the students have been polled.

a) Create a table of values that models this situation.

b) Does the data represent an exponential function? Explain.

7153_Ceng_M12_C6A_MCR_pp366-369.indd 368 posted in 11th pass on 12.07 12/7/11 2:02 PM

Lesson 6.3

7. Solve each equation. Explain the method you chose, and verify your solution by substitution.

a) $3^{2x} = 243$

b) $5^{x} = \frac{1}{125}$

c) $4^{x+2} = 64$

d) $6^{2x-1} = \frac{1}{216}$

8. The growth of a plant during a six-month period is modelled by the exponential function

$$f(x) = 3(1.25)^{x}$$

where $f(x)$ represents the height, in centimetres, at the end of each month and x represents the time in months.

a) Use a graph of the function to determine the height of the plant, to the nearest centimetre, at the end of the fourth month.

b) What was the initial height of the plant?

c) During which month was the plant double its original height?

9. Daphne has been recording the bounce heights of a ball. She determined that her data could be modelled by the exponential function

$$y = 140(0.80)^{x}$$

where y represents the height of the ball in centimetres and x represents the number of the bounce.

a) From what height was the ball first dropped?

b) Use a graph to determine the height of the ball, to the nearest centimetre, on the sixth bounce.

c) On which bounce was the height less than half the initial drop height?

10. A cup of coffee cools according to the function

$$C(t) = 90(0.977)^{t}$$

where $C(t)$ represents the temperature in degrees Celsius and t represents the time in minutes. The coffee will eventually reach a room temperature of 21 °C.

a) What was the initial temperature of the cup of coffee?

b) Determine the temperature of the cup of coffee after 5 min had elapsed.

c) Determine when the coffee will reach temperatures of 60 °C and 30 °C. Round your answers to the nearest minute.

11. Solve each equation. Explain the method you chose, and verify your solution by substitution.

a) $3(2)^{4x} = 192$

b) $3^{2-x} = \frac{1}{243}$

c) $4^{2x} = 8^{x+1}$

d) $2^{x-1} = 9^{x}$

7153_Ceng_M12_C6A_MCR_pp366-369.indd 369 posted in 11th pass on 12.07 12/7/11 2:02 PM

6.4 Modelling Data Using Exponential Functions

YOU WILL NEED

- large jar or plastic container
- pennies
- graphing technology

GOAL

Represent data using an exponential function, and interpret the graph to solve a problem.

INVESTIGATE the Math

Sabrina has been saving pennies in a jar. She decided to use these pennies to test an idea. She asked several of her friends to follow these steps:

- Take about 100 pennies from the jar, and place the pennies on a table.
- Put all the coins that have landed heads back in the jar.
- Pick up the remaining coins (the tails), shake them, and then place them on the table.
- Again, put all the coins that have landed heads back in the jar.
- Continue until only one coin or no coins are left.

Sabrina claimed that if all her friends graphed their results, all the graphs would look the same.

EXPLORE...

- A biologist was monitoring the growth in the number of fruit flies in a controlled environment.

The fruit fly population was recorded over a five-day period, as shown.

Day	Number of Flies
0	70
1	76
2	84
3	92
4	101
5	111

How can you estimate when the fruit fly population will exceed 160?

? What exponential function best models this situation?

A. Work in a group. Take a handful of pennies, and record the number in your handful. Predict how many pennies will land tails on the first toss. Record your data and predictions in a table like the one below.

Number of Tosses	Number of Tails	Prediction for Tails on Next Toss
0	initial sample size:	
1		
2		
3		

B. Shake the pennies, and then place them on a table. Remove all the pennies that landed heads, and count the remaining tails. Record the number of tails, and predict the number of coins that will land tails on the next toss. Repeat this process until your group has one penny left or runs out of coins.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 370 posted in 12th pass on 12.08 12/8/11 12:18 PM

C. Calculate the ratios for consecutive pairs of numbers in the last column of your table. What do you notice? Explain.

D. Should an increasing exponential function or a decreasing exponential function be used to model the data you collected? Explain.

E. Create a scatter plot for "Number of Tails" versus "Number of Tosses."

F. Perform an exponential regression on your data to determine the curve of best fit and its equation.

Reflecting

G. Sabrina claimed that if all her friends graphed their results, all the graphs would look the same. Do you agree with Sabrina? How is your graph the same as the other groups' graphs? How is it different?

H. How do the parameters in the exponential equation $y = a(b)^x$ relate to the parameters of the equation of your regression function?

I. What are the domain and range of your function in this context? Explain.

APPLY the Math

EXAMPLE 1 Creating graphical and algebraic models of given data

The population of Canada from 1871 to 1971 is shown in the table below. In the third column, the values have been rounded.

Year	Actual Population of Canada	Population of Canada (millions)
1871	2 436 297	2.40
1881	3 229 633	3.23
1891	3 737 257	3.74
1901	5 418 663	5.42
1911	7 221 662	7.22
1921	8 800 249	8.80
1931	10 376 379	10.38
1941	11 506 655	11.51
1951	14 009 429	14.01
1961	18 238 247	18.24
1971	21 568 305	21.57

Statistics Canada

Top: Main St., Winnipeg, in 1880.
Bottom: Main St., Winnipeg, in 2010.

a) Using graphing technology, create a graphical model and an algebraic exponential model for the data.

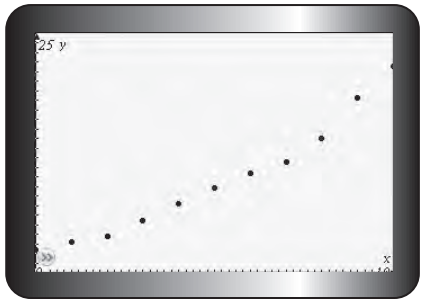
b) Assuming that the population growth continued at the same rate to 2011, estimate the population in 2011. Round your answer to the nearest million.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 371 posted in 11th pass on 12.07 12/7/11 2:03 PM

Luba's Solution

a) Let x represent the number of 10-year intervals since 1871. Let y represent the population of Canada in millions.

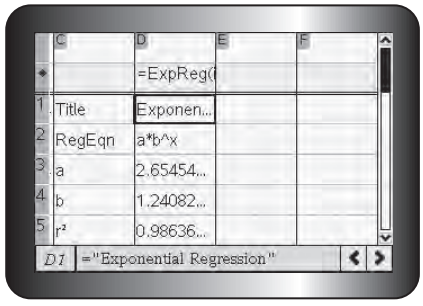
I defined variables x and y.



I entered the numbers 0 to 10, to represent the years 1871 to 1971, into list one of my graphing calculator. I entered the data from the third column of the data table into list two of my graphing calculator. Then I graphed the points in a scatter plot.

I noticed that the y-values grow as x gets larger, but not at a constant rate. I predicted that the data could be modelled by an **exponential growth function.**

I used my calculator to do an exponential regression on the data, to determine the equation of the curve of best fit.

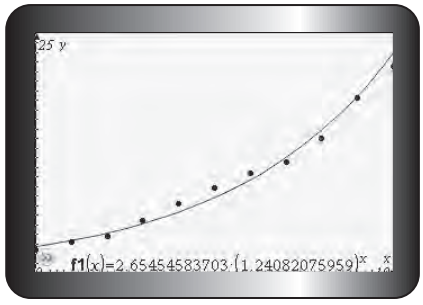


exponential growth function

An exponential function whose y-values increase as you move from left to right along the x-axis; for an exponential function of the form $y = a(b)^x$, exponential growth occurs when $a > 0$ and $b > 1$.

The values of a and b agree with my prediction that the model is an exponential growth function, since $a > 0$ and $b > 1$.

I verified the equation by graphing it on the same grid as the given points.



I noticed that even though some points lie above or below the graph, the exponential regression function appears to be a good model for the data.

$y = a(b)^x$

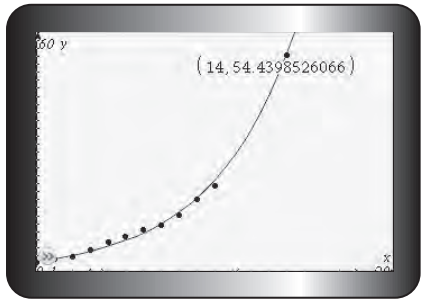
The exponential equation that models the data is

$y = 2.654...(1.240...)$

I used the regression result to determine the values of a and b, $a = 2.654...$ and $b = 1.240...$.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 372 posted in 12th pass on 12.08 12/8/11 12:18 PM

b) $x = \dfrac{2011 - 1871}{10}$ or 14



To determine the population in 2011, I first needed to determine the number of 10-year intervals from 1871 to 2011.

I extrapolated the *y*-value from the graph at $x = 14$.

$y \doteq 54.44$

Verify:

$y = 2.654...(1.240...)^x$

$y = 2.654...(1.240...)^{14}$

$y = 54.439...$

I verified my extrapolated value by substituting $x = 14$ into the regression equation. My result is the same as the value extrapolated from the graph.

Assuming that the population growth continued at the same rate, the population of Canada would be about 54 million in 2011.

Your Turn

a) Determine an exponential model for the data in the original problem, if *x* represents the actual year and *y* represents the actual population.
b) How does your model differ from Luba's model?
c) Does your model predict the same population of Canada in 2011? Explain.
d) Which model do you prefer? Explain why.

EXAMPLE 2 Solving a problem using an exponential regression model

Sonja did an experiment to determine the cooling curve of water. She placed the same volume of hot water in three identical cups. Then she recorded the temperature of the water in each cup as it cooled over time. Her data for three trials is given as follows.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 373 posted in 11th pass on 12.07 12/7/11 3:56 PM

Trial 1

Time (min)	Temperature (°C)
0	80
5	69
10	61
20	45
30	34
40	26

Trial 2

Time (min)	Temperature (°C)
0	75
5	66
10	59
20	44
30	32
40	23

Trial 3

Time (min)	Temperature (°C)
0	78
5	68
10	61
20	44
30	33
40	25

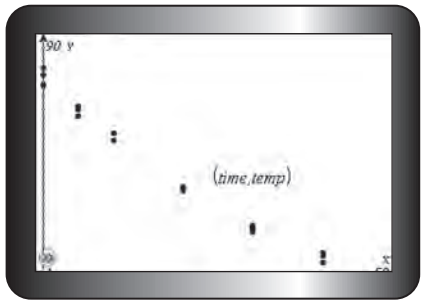
a) Construct a scatter plot to display the data. Determine the equation of the exponential regression function that models Sonja's data.

b) Estimate the temperature of the water 15 min after the experiment began. Round your answer to the nearest degree.

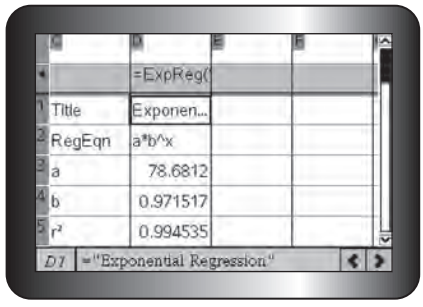
c) Estimate when the water reached a temperature of 30 °C. Round your answer to the nearest minute.

Clare's Solution

a) Let x represent the time, in minutes, since the experiment began.
Let y represent the temperature in degrees Celsius.



I used my calculator to do an exponential regression on the data, to determine the equation of the curve of best fit.



I defined variables for the data. Then I entered the data for all three trials into my graphing calculator and created a scatter plot.

Since the points are decreasing, I knew that the data can be modelled by an **exponential decay function.**

exponential decay function

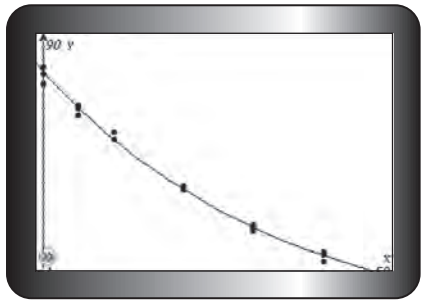
An exponential function whose y-values decrease as you move from left to right along the x-axis; for an exponential function of the form $y = a(b)^x$, exponential decay occurs when $a > 0$ and $0 < b < 1$.

The values of a and b agree with my prediction that the model is an exponential decay function, because $a > 0$ and $0 < b < 1$.

The initial temperatures of the three samples were not the same: 80 °C, 75 °C, and 78 °C. The regression model defines $a = 78.68$, which is close to all three initial values.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 374 posted in 11th pass on 12.07 12/7/11 2:04 PM

I verified the regression function by graphing it on the same grid as the given data.

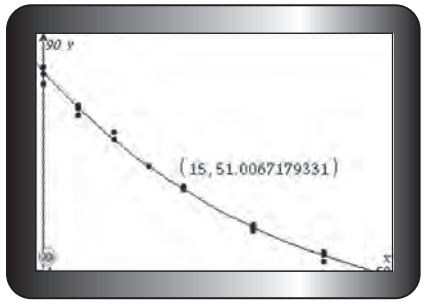


Even though some data points lie above or below the graph, the regression function appears to be a good model for the data.

The equation of the exponential function that models the given data is

$y = 78.681...(0.971...)^x$

b)



To determine the temperature of the water at 15 min, I interpolated the y-value from the graph at $x = 15$.

About 15 min after the experiment started, the temperature of the water should be about 51 °C.

c) Substitute $y = 30$ into the exponential regression model.

$y = 78.681...(0.971...)^x$

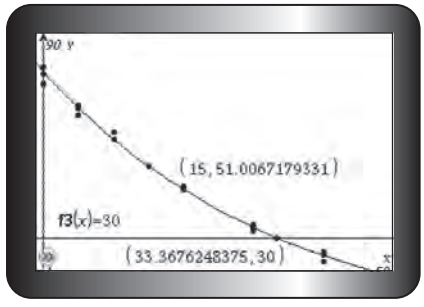
$30 = 78.681...(0.971...)^x$

To determine the time when the water reached a temperature of 30 °C, I had to solve the exponential equation. I realized that I could not solve the equation algebraically, so I decided to use graphing technology.

$y_1 = 78.681...(0.971...)^x$

$y_2 = 30$

I knew that I could use the system of equations and point of intersection strategy I have used before with polynomial functions to solve this equation. This enables me to determine the value of the independent variable, x, when I know the value of the dependent variable, y, in any type of equation.



I already had the graph of the curve of best fit, so I entered $y = 30$ into my calculator to create the linear exponential system of equations that corresponded to the equation $30 = 78.681...(0.972...)^x$.

The point of intersection is (33.367..., 30).

$x = 33.367...$

The x-coordinate of this point is the solution to the equation $30 = 78.681...(0.972...)^x$.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 375 posted in 12th pass on 12.08 12/8/11 12:18 PM

After a little over 33 min from the start of the experiment, the water should reach a temperature of 30 °C.

Your Turn

Emma did the same experiment, but performed only one trial. Her data is given below.

Time (min)	0	5	10	20	30	45
Temperature (°C)	90	78	68	52	38	26

a) Construct a scatter plot to display Emma's data. Determine the equation of the exponential regression function that models her data.

b) Compare the characteristics of the exponential regression function for Emma's data with the characteristics of the function for Sonja's data. How are the graphs the same? How are they different?

c) At what time, to the nearest minute, did the water reach 51 °C in Emma's experiment? How much longer did it take than in Sonja's experiment?

In Summary

Key Idea

- Graphing technology can be used to create a scatter plot. When the scatter plot looks like an exponential function, exponential regression can be used to determine an algebraic model for the data.
- An exponential function models growth when $a > 0$ and $b > 1$. The y-values of an exponential growth function increase as you move from left to right along the x-axis.
- An exponential function models decay when $a > 0$ and $0 < b < 1$. The y-values of an exponential decay function decrease as you move from left to right along the x-axis.

Need to Know

- The exponential regression model is
$$f(x) = a(b)^x$$
where a represents the initial value, and b represents the growth factor if $b > 1$ or the decay factor if $0 < b < 1$.
- An exponential curve of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the curve of best fit on a scatter plot or by using the equation of the exponential regression function.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 376 posted in 11th pass on 12.07 12/7/11 2:04 PM

CHECK Your Understanding

1. Determine which of the following data sets involve exponential growth or decay. Explain how you know. Then graph each data set that involves exponential growth or decay to verify your answer, determine the equation of the exponential regression function, and graph the function on the same grid as the data set.

a)

x	*y*
0	0.0
1	5.1
2	10.2
3	15.3
4	20.4
5	25.5

b)

x	*y*
0	1.00
1	2.50
2	6.25
3	15.62
4	39.06
5	97.66

c)

x	*y*
−3	−2.25
−2	−1.00
−1	−0.25
0	0.00
1	−0.25
2	−1.00

d)

x	*y*
2	32
3	108
4	256
5	500
6	864
7	1372

2. Use exponential regression to determine the equation of the curve of best fit for each data set. Then state the characteristics of each function, including the domain and range.

a)

x	*y*
−3	1255.05
−2	248.50
−1	49.58
0	10.30
1	2.10
2	0.40
3	0.08

b)

x	*y*
1	2.93
2	3.09
3	3.26
4	3.44
5	3.63
6	3.83
7	4.02

PRACTISING

3. Four years ago, the Belands retired and decided to move into an apartment. Their annual rent for the four years is given below.

Years since Retirement	0	1	2	3
Rent ($)	9600	9960	10 344	10 752

a) Determine if the data can be represented by an exponential model. How do you know?

b) Explain how you can use the annual rate of increase in the rent to define an equation that models the data.

c) Assuming the same annual rate of increase, predict what the Belands will pay in 10 years. Round your answer to the nearest dollar.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 377 posted in 11th pass on 12.07 12/7/11 2:04 PM

4. Samuel and his grandfather recorded the length and mass of every rainbow trout they caught in the Red Deer River during the summer. Samuel plans to use the data for an upcoming science fair project.

Length (mm)	270	392	360	335	259	324	317	351	309	414	388	440	266	338
Mass (g)	210	622	480	474	203	357	338	504	392	750	661	843	225	459

a) Construct a scatter plot to display the data.
b) Determine the equation of the exponential regression function for the data. Graph the function on the same grid that you plotted the points.
c) Samuel caught a fish that measured 400 mm. Use your graph to estimate the mass of the fish, to the nearest gram. Describe your process.
d) Samuel's grandfather would like to catch a fish with a mass of 1000 g. How long, to the nearest millimetre, would you expect this fish to be? Describe your process.

5. The population of the world has been growing exponentially for about 150 years. The data table shows the global population in billions over a 40-year period.

Year	1970	1980	1990	2000	2010
Population (billions)	3.91	4.52	5.22	6.03	6.97

a) Construct a scatter plot to display the data.
b) Use exponential regression to define a function that models the data.
c) Assuming the same growth rate, estimate the population in the year 2020, to two decimal places. Describe your process.
d) If the growth rate remains the same, when would you expect the population to reach 9.50 billion? Describe your process.

6. The height of a sunflower was recorded every seven days as it grew.

Day	7	14	21	28	35	42
Height (cm)	17.9	36.4	67.8	98.0	131.1	169.7

a) Use exponential regression to model the growth of the sunflower.
b) Estimate the height of the sunflower, to the nearest tenth of a centimetre, on day 10.
c) Estimate the height of the sunflower, to the nearest tenth of a centimetre, on day 30. Does your answer make sense, based on the data? Explain.
d) On what day would you expect the sunflower to reach a height of 50 cm?

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 378 posted in 11th pass on 12.07 12/7/11 2:04 PM

7. Computer chips use binary codes to operate. These codes are made up of binary numbers, which are combinations of zeros and ones. A binary number is a number written with a base of 2.

a) The following table shows the binary codes that can be made with one to four digits. Determine the equation of the exponential function that models the growth in the number of possible codes. Describe your process.

Number of Digits	Possible Codes	Number of Possible Codes
1	0, 1	2
2	00, 01, 10, 11	4
3	000, 001, 010, 011 100, 101, 110, 111	8
4	0000, 0001, 0010, 0011 0100, 0101, 0110, 0111 1111, 1110, 1101, 1100, 1011, 1010, 1001, 1000	16

b) State the domain and range of your function.

c) How many digits are needed to have a code for each of the 104 keys on a standard QWERTY keyboard?

8. Air pressure around a plane decreases as the plane's altitude increases. The data at the right shows the pressure, in millibars, that is exerted by the atmosphere on an object at different altitudes, in kilometres.

Altitude (km)	Air Pressure (mb)
1	898.7
2	795.0
3	701.2
4	616.0
5	540.5
6	472.2
7	411.1
8	356.5
9	307.0
10	264.9

a) Create a scatter plot to display the data.

b) Determine the equation of the exponential regression function that models the data. Graph the function on the same grid as your scatter plot.

c) Estimate the pressure at 15 km, to the nearest tenth of a millibar.

d) Estimate the altitude, to the nearest kilometre, where the pressure reaches 500.0 mb and 50.0 mb.

9. A biologist has been studying the effects of environmental stressors on a population of frogs. The observations are shown in the table.

Time (years)	0	1	2	3	4	5
Frog Population	575	460	368	294	235	188

a) By what percent is the frog population declining every year?

b) Determine the equation of the exponential regression function that models the decline in the frog population.

c) The biologist determines that the half-life of the frog population is about 3 years. Estimate the half-life by interpolation, and then use your regression equation to verify your estimate algebraically.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 379 posted in 11th pass on 12.07 12/7/11 2:04 PM

10. The Mauna Loa observatory in Hawaii has been collecting monthly data on the proportion of carbon dioxide (CO_2) in the atmosphere, in parts per million, since 1959. The data that was collected at the end of the year, every five years from 1960 to 2005, is shown in the table below.

Year	Atmospheric CO_2 (ppm)
1960	319
1965	322
1970	328
1975	333
1980	341
1985	348
1990	356
1995	364
2000	372
2005	381

a) Could a linear function model the data? Justify your answer.
b) Determine the equation of the exponential function that models the data.
c) Verify your model by doing an exponential regression.
d) According to your model, what is the predicted proportion of atmospheric CO_2 in 2010 and 2020, rounded to the nearest part per million?

11. Sebastian prepares hot sandwiches at his father's restaurant. He leaves the sandwiches on the counter to cool slightly before they are served. To prevent customers from burning their mouths, he conducts a test to determine how long he should leave the sandwiches to cool. He measures the internal temperature of a hot sandwich as it is cooling. His data is shown below.

Time (min)	0	1	2	3	4	5
Temperature (°C)	84.0	79.8	75.8	72.7	68.7	65.5

a) Determine the equation of the exponential regression function that models this situation.
b) How long, to the nearest minute, will it take for the sandwich to have an internal temperature that is half of its initial temperature?
c) How long, to the nearest minute, will the sandwich need to sit before its internal temperature is the same as the room temperature, 20 °C?

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 380 posted in 11th pass on 12.07 12/7/11 2:04 PM

12. Tristan drank a cup of tea that contained 40 mg of caffeine. The amount of caffeine in his body was measured repeatedly after drinking the tea.

Time (h)	0	1	3	4	6
Amount of Caffeine (mg)	40	31	17	13	7

a) Using graphing technology, create a graphical exponential model and an algebraic exponential model for the data.
b) When will Tristan have less than 3 mg of caffeine in his body? Round your answer to the nearest hour.
c) Determine the half-life of the caffeine in Tristan's body. Round your answer to the nearest unit.

13. A sample of bacteria was taken from a patient. A lab technician then recorded the growth of the bacterial population at fixed time intervals. The first five observations are given below.

Observation	1	2	3	4	5
Number of Bacteria	6180	9270	13 905	20 857	31 285

a) Determine the equation of the exponential regression function that models the data.
b) If this growth rate continues, how many bacteria will there be when the lab technician makes the 10th observation?
c) A drug has been developed that affects the growth rate of the bacterial population. When studying a second identical sample that has been treated with the drug, the lab technician notices that the population now grows at a rate of 20% per hour. What function models the growth of the population in the second sample?

14. Researchers have been studying algae growth on small lakes and ponds in southern Manitoba. They measured the percentage of the surface of a pond that was covered by algae at fixed time intervals, as shown below.

Week	0	1	2	3	4	5
Pond Surface Covered by Algae (%)	25.0	27.5	30.3	33.3	36.6	40.3

a) By what percent are the algae growing every week?
b) Assuming the same growth rate, how many weeks will it take for the algae to cover 80% of the pond?

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 381 posted in 11th pass on 12.07 12/7/11 2:04 PM

15. Graham, the lead geologist for an oil and gas company, has been analyzing data taken from a proposed drill site in southern Saskatchewan. He has determined that a well at this site should produce 200 barrels of oil per week over the next 10 years. At that point, however, the production of the well will begin to drop, according to the exponential regression function

$$P(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{48}}$$

where $P(t)$ represents the weekly amount of oil produced, in barrels, and t represents the number of weeks after production begins to drop.

a) Determine the expected production of this well, to the nearest barrel per week,
   i) 26 weeks after production begins to drop
   ii) 100 weeks after production begins to drop

b) The well will require mechanical assistance to bring up the oil once production drops to 30 barrels per week. During which week will production reach this level?

c) In his report, Graham has informed the company that the well will no longer be profitable once production drops to 5 barrels per week. During which week should the company cap the well?

16. The output of an oil well in Saskachewan is initially 3000 barrels per week. Each week that the well operates, its output decreases by 10%.

a) Determine the equation of the exponential function that models the output of the well.

b) Assuming the same decrease in output, how much oil will be extracted from the well at the end of 20 weeks of operation?

c) The oil company will close the well when its output drops to less than five barrels per week. Assuming the same decrease in output, during which week will the well be closed?

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 382 posted in 11th pass on 12.07 12/7/11 2:04 PM

Closing

17. The light intensity of a spotlight can be reduced by using gels, which are placed in front of the light. The table to the right represents the intensity of a spotlight after a specific number of gels have been used.

a) Explain how you could create an algebraic model of the data.

b) Determine the light intensity of a spotlight if seven gels are used, to two decimal places.

c) What is the minimum number of gels that would be needed to reduce the light intensity by more than 60.00%?

Number of Gels Used	Intensity (%)
0	100.00
1	97.00
2	94.09
3	91.27
4	88.53

Extending

18. Melinda used toothpicks to create a pattern of figures. The first three figures are shown below. She claims that this pattern can be modelled by an exponential function.

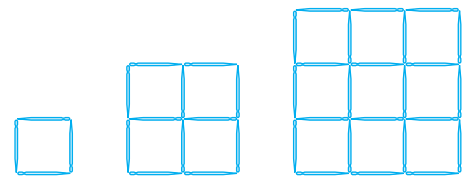


figure 1 figure 2 figure 3

a) How many toothpicks will be needed to create the next figure in the pattern?

b) Is Melinda's claim true? Explain.

c) Assuming that this pattern continues, how many toothpicks will be needed to make the 7th figure?

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 383 posted in 11th pass on 12.07 12/7/11 3:56 PM

19. A 100.0 g sample of radioactive tin-117 decayed to 16 g in 37 days, as shown below.

Number of Days	0	9	15	24	37
Mass of Tin-117 (g)	100.0	64.0	47.6	30.5	16.0

The function that models this decay can be written as

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where $A(t)$ represents the mass of tin-117, A_0 represents the initial mass of tin-117, t represents the time in days, and h represents the time needed for half of the tin-117 to decay.

a) Use one of the points in the table to determine the value of h.

b) Use your answer from part a) to write a more precise exponential function that models the decay of tin-117.

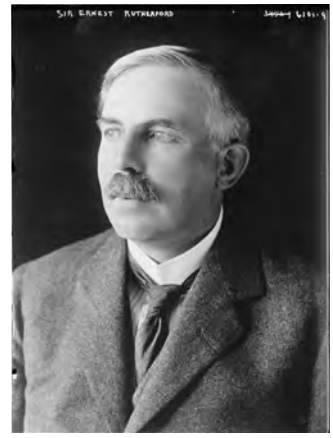
c) Determine the equation of the exponential regression function that models the decay of tin-117.

d) Verify that your equation from part b) matches your regression equation by using graphing technology.

History | Connection

Ernest Rutherford (1871–1937)

Many people consider Ernest Rutherford to be one of most important nuclear physicists. He was born in Nelson, New Zealand, and eventually came to Canada in 1898. He held the position of the MacDonald Chair of Physics at McGill University in Montreal. His work on the structure of the atom involved examining the decay of radioactive substances. He found that various particles are emitted as a radioactive element decays into different elements. The alpha particle, beta particle, and proton were named and described by him. Rutherford won the Nobel Prize in Chemistry in 1908.



Ernest Rutherford

A. Do research to determine the radioactive elements that Rutherford used in his work.

B. How could exponential functions be used to analyze the radioactive decay of an element? Provide exponential functions that could model the elements that Rutherford used.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 384 posted in 11th pass on 12.07 12/7/11 2:04 PM

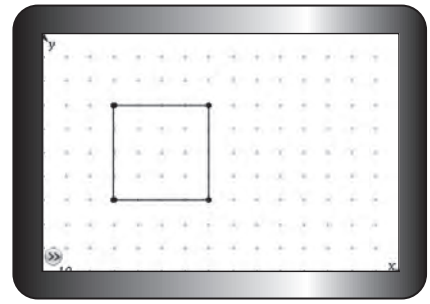
Applying Problem-Solving Strategies

Exponential Hit

YOU WILL NEED

- graphing calculator

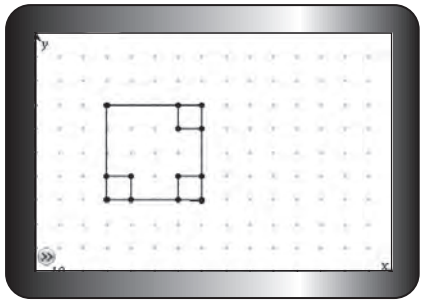
Create an exponential function so that the function passes through an opponent's square. Players win by discovering all of their opponent's squares.

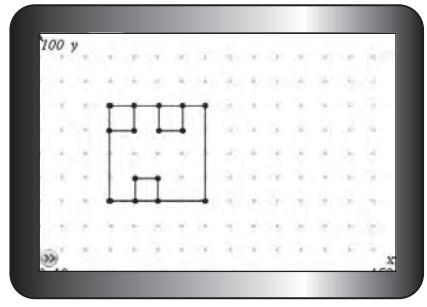


The Game

A. Both players need to construct the same playing grid on their graphing calculators; for example, a grid with the coordinates (30, 30), (30, 70), (70, 30), and (70, 70). Both players also need to have their calculators set to the same window.

B. Each player secretly draws three boxes in the grid. Two examples are presented below.





C. Players decide if they will allow only increasing exponential functions, only decreasing exponential functions, or both.

D. Players decide who will go first, using "rock, paper, scissors" or a similar technique.

E. One player calls out an exponential function. The other player enters this function into his or her calculator and determines if the function intersects any of his or her own squares. If it does, the player must declare that the square has been "hit." If it does not, the player who graphed the function must declare how the function missed the square by using words such as left, right, above, or below.

F. Players take turns calling out functions. The player with one or more squares remaining, when all the other player's squares have been hit, is the winner.

The Strategy

G. Suggest some characteristics of functions that are more likely to be hit. Are there locations that make squares less likely to be hit?

Variation

H. Suggest some variations of the rules to improve the game.

7153_Ceng_M12_C6A_L6.4_pp370-385.indd 385 posted in 11th pass on 12.07 12/7/11 3:56 PM

6.5 Financial Applications Involving Exponential Functions

YOU WILL NEED

- graphing calculator or spreadsheet

GOAL

Solve problems that involve loans and mortgages by using exponential functions.

EXPLORE...

- Erica has been using these exponential growth functions to model various financial situations.

  $y = 1000(1.03)^x$
  $y = 1000(1.06)^x$
  $y = 1000(1.09)^x$

  She has noticed a relationship between the time it takes for her investments to double in value, x, and the interest rate paid on her investments. How long, in years, would it take for each of her investments of $1000 to double in value?

compound interest

The interest earned on both the original amount that was invested and any interest that has accumulated over time.

Communication Tip

Interest rates are usually expressed as a percent per annum (/a). For example, an interest rate of 5%/a means that interest will be paid (or charged) at a rate of 5% each year.

INVESTIGATE the Math

The cost of living is affected, in part, by inflation. Suppose that the rate of inflation is 2% over the preceding year. This means that if you spent $1000 on goods and services a year ago, then the same goods and services would now cost 2% more, or $1020.

Arthur wants to stay ahead of the rate of inflation, so he decides to invest $1000 in a financial product that is paying more than 2% interest. His financial advisor presents him with three different products that earn **compound interest**. The compound interest is calculated at the end of each year at the rates of 3%/a, 4%/a, and 6%/a, respectively.

? How many years will it take for Arthur's $1000 to double in value at each compound interest rate?

A. Calculate the value of Arthur's $1000 investment at 3% compound interest over 6 years. Record your calculations in a table like the one below.

3% Compound Interest per Year		
Year	**Starting Value**	**Ending Value**
0	$1000.00	$1000.00 + 0.03($1000.00) = $1030.00
1	$1030.00	$1030.00 + 0.03($1030.00) = $1060.90
2	$1060.90	$1060.90 + 0.03($1060.90) = $1092.73
3		

B. Explain how you can calculate the value at the end of each year in one step, instead of adding two terms.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 386 posted in 11th pass on 12.07 12/7/11 2:03 PM

C. Create a scatter plot using the time in years and the starting value in dollars as the variables.

D. Use exponential regression to determine the equation of the exponential regression function that models the data.

E. Determine the time needed to double Arthur's initial investment.

F. Repeat steps A to E at 4% and 6% compound interest over 6 years.

Reflecting

G. Examine your data and graph for each of the three models. How do the models relate to the compound interest formula

$$A(n) = P(1 + i)^n$$

where $A(n)$ represents the **future value**, P represents the **principal**, i represents the interest rate per **compounding period**, and n represents the number of compounding periods?

H. Verify that the compound interest formula produces an exponential equation for each of the compounding rates in the problem.

I. In this context, was the data continuous or discrete? Explain.

future value
The amount that an investment will be worth after a specified period of time.

principal
The original amount of money that is invested or borrowed.

compounding period
The time over which interest is calculated and paid on an investment or loan.

APPLY the Math

EXAMPLE 1 Calculating the time for an investment to grow to a certain amount

Brittany invested $2500 in an account that pays 3.5%/a compounded monthly. The following table gives the value of her investment for the first five months.

Value of Investment ($)	2500.00	2507.29	2514.60	2521.94	2529.29	2536.67
Time (months)	0	1	2	3	4	5

a) Use exponential regression to determine the compound interest function that models this situation.
b) Explain how the values in the table were determined.
c) How long, in months, will it take for Brittany's investment to grow to $3000?

Communication *Tip*
Some common compounding periods are daily, weekly, monthly, quarterly, semi-annually, and annually. If the compounding period is monthly, interest is calculated at the end of every month and added to the balance. The new total becomes the starting value for the next month.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 387 posted in 11th pass on 12.07 12/7/11 2:03 PM

Josh's Solution

a) Interest rate: 3.5%/a compounded monthly
Interest rate per compounding period
$= \frac{0.035}{12}$ or 0.002...

I started by determining the compound interest equation that models this situation.

I noticed that the given interest rate is "per year." I also noticed that the interest is compounded monthly.

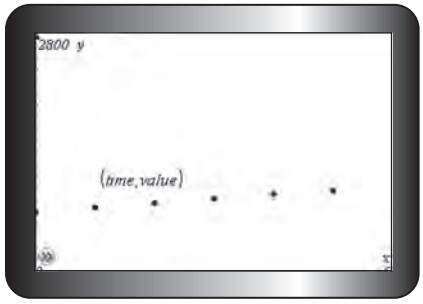
I needed to calculate the interest rate per compounding period. To do this, I divided the annual interest rate, 3.5%, by 12, the number of months in a year.

$P = 2500$
$i = 0.002...$
n, the number of compounding periods, is unknown
$A = 2500(1 + 0.002...)^n$

I substituted the principal and the interest rate per compounding period into the compound interest formula to get the compound interest function that models this situation.

The number of compounding periods, n, is the value I had to determine.

To verify the compound interest function, I created a scatter plot by entering the given data into my calculator.



The scatter plot looks like a straight line.

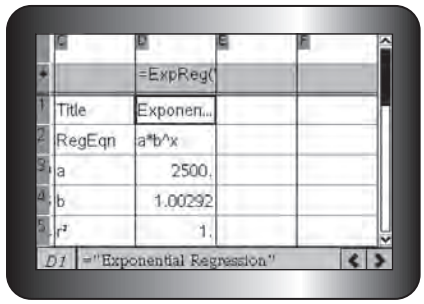
To verify that the relation is not linear, I decided to compare the differences in the monthly values.

$2507.29 - 2500.00 = 7.29$
$2514.60 - 2507.29 = 7.31$
$2521.94 - 2514.60 = 7.34$
$2529.29 - 2521.94 = 7.35$
$2536.67 - 2529.29 = 7.38$

The amount for each month gains a few more interest dollars than the amount for the previous month. This means that the graph is not linear.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 388 posted in 11th pass on 12.07 12/7/11 2:03 PM

I used my calculator to determine the equation of the exponential regression function that models this data.



$y = 2500(1.002...)^x$

The regression function is the same as the compound interest equation I determined earlier.

I verified that the graph of the exponential regression function includes all the data points.

b) Value after 1st month:

$A = 2500(1 + 0.002...)^1$
$A = 2507.29$

Value after 2nd month:

$A = 2500(1 + 0.002...)^2$
$A = 2514.60$

Value after 3rd month:

$A = 2500(1 + 0.002...)^3$
$A = 2521.94$

Value after 4th month:

$A = 2500(1 + 0.002...)^4$
$A = 2529.29$

Value after 5th month:

$A = 2500(1 + 0.002...)^5$
$A = 2536.67$

To duplicate the values in the table, I applied the compound interest formula:

$$A(n) = P(1 + i)^n$$

Since the value of Brittany's investment after the first 5 months was given, I substituted $n = 1, 2, 3, 4$, and 5.

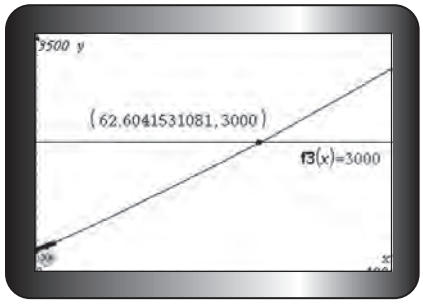
7153_Ceng_M12_C6A_L6.5_pp386-399.indd 389 posted in 11th pass on 12.07 12/7/11 2:03 PM

c) $3000 = 2500(1.002\,917)^x$

To determine when the future value will grow to \$3000, I substituted 3000 for y in the exponential regression function. This resulted in an exponential equation.

$y_1 = 3000$
$y_2 = 2500(1.002\,917)^x$

I couldn't write both sides of this equation as powers with like bases so I used a system of equations strategy. I created linear function y_1, using the left side of the equation, and exponential function y_2, using the right side of the equation.



I graphed both functions, then determined that the point of intersection was (62.604..., 3000).

$x = 62.604...$
It will take about 63 months for Brittany's investment to grow to \$3000.

The x-coordinate of the point of intersection is the solution to the equation

$$3000 = 2500(1.002\,917)^x$$

Your Turn

After Brittany's investment reached \$3000, her bank changed the interest rate to 3.5%/a, compounded semi-annually. Brittany decided to reinvest at this new rate.

a) How much will her investment be worth at the end of each of the next 4 years?

b) How long, in years, will it take her investment to reach \$5000?

EXAMPLE 2 Modelling depreciation with an exponential function

Gina bought a camera for her studio 2 years ago. Her accountant told her that, starting from the beginning of the second year, she can claim a **depreciation** rate of 20% for the camera as a business expense. At the beginning of the second year, Gina's camera was worth \$2000.

depreciation
The decrease in the value of an asset over time.

a) How long, in years from the time of purchase, will it take for the camera to be worth only \$200?

b) Explain how the exponential regression function that models this situation relates to the depreciation of the camera.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 390 posted in 11th pass on 12.07 12/7/11 2:03 PM

Gina's Solution

a)

Years After Purchase	Value at Start of Year ($)	Depreciation ($)	Value at End of Year ($)
2	2000	$2000 \cdot 20\% = 400$	$2000 - 400 = 1600$
3	1600	$1600 \cdot 20\% = 320$	$1600 - 320 = 1280$
4	1280	$1280 \cdot 20\% = 256$	$1280 - 256 = 1024$
5	1024	$1024 \cdot 20\% = 204.80$	$1024 - 204.80 = 819.20$
6	819.20	$819.20 \cdot 20\% = 163.84$	$819.20 - 163.84 = 655.36$
7	655.36	$655.36 \cdot 20\% = 131.07$	$655.36 - 131.07 = 524.29$

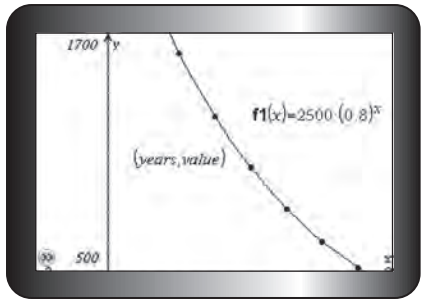
I made a table of values for the depreciation of the camera, starting with the second year after purchase. I noticed that the depreciation each year is not the same. This means that I can claim less each year as a business expense.

Let x represent the number of years after purchase, and let y represent the value of the camera:

$y = 2500(0.8)^x$

I defined the variables for this situation.

I entered the values from the "Years After Purchase" column and the "Value at End of Year" column into my calculator. Then I determined the equation of the exponential regression function that models the data.



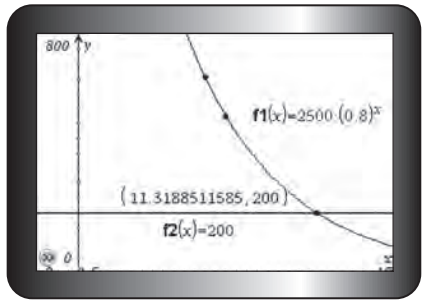
I verified that the graph of the exponential regression function includes the points in my table of values.

$200 = 2500(0.8)^x$

To determine when the camera will be worth $200, I substituted 200 for y in the exponential regression function. This resulted in an exponential equation.

$y_1 = 200$
$y_2 = 2500(0.8)^x$

I couldn't write both sides of this equation as powers with like bases so I decided to use a system of equations strategy. I created linear function y_1, using the left side of the equation, and exponential function y_2, using the right side of the equation.



I graphed both functions, then determined that the point of intersection was (11.318..., 200).

$x = 11.318...$

The x-coordinate of the point of intersection is the solution to the equation

$200 = 2500(0.8)^x$

My camera will be worth $200 at the end of the 12th year after purchase.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 391 posted in 11th pass on 12.07 12/7/11 2:03 PM

b) The exponential regression function is $y = 2500(0.8)^x$.
The base of the regression function is 0.8 or 80%.
This makes sense because the camera loses 20% of its value each year:

$$100\% - 20\% = 80\%$$

Since y is the value of the camera, the camera is worth 80% of its value from the previous year.

The multiplier of the regression equation is 2500.
The value of the camera was \$2000 at the start of the second year, but its value was greater at purchase.

Your Turn

Ali inherited a set of rare silver coins from his great-grandfather. An appraiser told Ali that the coins are currently worth \$1000 and their **appreciation** rate will likely be 2.5%/a.

a) Determine the equation of the exponential regression function that models this situation.

b) How long, in years from the time of appraisal, will it take for the coins to be worth \$2000?

appreciation
The increase in the value of an asset over time.

EXAMPLE 3 Calculating the time to pay off a loan

Jessica borrowed \$7500 from a bank to buy new equipment for her band. The bank is charging an interest rate of 3.6%/a compounded monthly. Jessica's monthly loan payment is \$400.

a) Determine how long, to the nearest month, it will take Jessica to pay off the loan. The loan manager gave Jessica the following equation so she could determine how long it would take to pay off her loan

$$(1.003)^{-n} = 0.943\,75$$

where n represents the number of months.

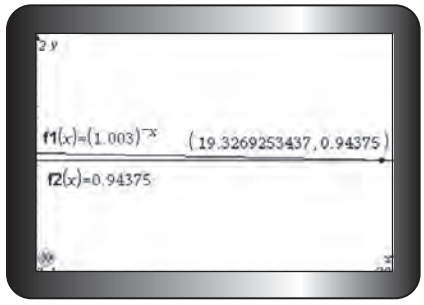
b) How much interest will Jessica pay on the loan?

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 392 posted in 11th pass on 12.07 12/7/11 2:03 PM

Jessica's Solution

a) $(1.003)^{-n} = 0.94375$

$y_1 = (1.003)^{-n}$
$y_2 = 0.943...$



$x = 19.326...$

I will pay off the loan on the 20th month.

I could not write both sides of the equation as powers with like bases, so I used a system of equations strategy. I graphed both sides of the equation as functions on my graphing calculator.

I determined that the point of intersection was (19.326..., 0.943 75). The x-coordinate of this point represents the number of months it will take to pay off the loan.

The last full payment will be in the 19th month. The payment in the 20th month will be only a fraction of the regular monthly payment.

b) Total of loan payments = (\$400)(19.326...)
Total of loan payments = \$7730.77

I multiplied the monthly payment by the number of months it will take to pay off the loan.

Interest = (Total of loan payments) − (Amount borrowed)
Interest = \$7730.77 − \$7500.00
Interest = \$230.77

To determine the interest charged, I subtracted the amount that I borrowed from the amount that I had to repay.

I will pay \$230.77 in interest on the loan.

Your Turn

Jessica is thinking about adding a lighting and sound package to her purchase. This would add \$3000 to her loan.

a) How much is Jessica borrowing now?

b) The loan manager gave Jessica the following equation so she could determine how long it would take to pay off her loan

$$(1.003)^{-n} = 0.921\ 25$$

where n represents the number of months. How much longer will it take to pay off the loan?

c) What is the total amount of interest that Jessica will pay on the loan now? Explain.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 393 posted in 11th pass on 12.07 12/7/11 2:03 PM

In Summary

Key Ideas

- Financial institutions charge compound interest on all loans and pay compound interest on some investments. Compound interest is calculated at regular intervals by first adding any accumulated interest to the principal and then multiplying the result by the interest rate.
- When money earns or is charged compound interest, an exponential function can be used to model the situation.

Need to Know

- The exponential function that models compound interest is

$$A(n) = P(1 + i)^n$$

where A represents the future value; P represents the principal; i represents the interest rate per compounding period, expressed as a decimal; and n represents the number of compounding periods.

- Compounding periods are usually daily, weekly, monthly, quarterly, semi-annually, or annually. The following table shows how the interest rate per compounding period is determined.

Compounding Period	Number of Times Interest Is Paid	Interest Rate per Compounding Period, i
daily	365 times per year	$i = \frac{\text{annual rate}}{365}$
weekly	52 times per year	$i = \frac{\text{annual rate}}{52}$
semi-monthly	24 times per year	$i = \frac{\text{annual rate}}{24}$
monthly	12 times per year	$i = \frac{\text{annual rate}}{12}$
quarterly	4 times per year	$i = \frac{\text{annual rate}}{4}$
semi-annually	2 times per year	$i = \frac{\text{annual rate}}{2}$
annually	1 time per year	$i = \frac{\text{annual rate}}{1}$

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 394 posted in 11th pass on 12.07 12/7/11 2:03 PM

CHECK Your Understanding

1. Danielle invested money in two different accounts. Her investments can be modelled by the following growth functions, where x represents the time in years:

A. $y = 1500(1.05)^x$

B. $y = 2000(1.035)^x$

a) What was the principal that Danielle invested in each account?

b) What was the annual interest rate for each investment? Explain.

c) Determine the value of each investment at the end of 5 years.

d) How long, in months, did it take for each account to have $3000?

2. Principal of $3000 was invested for 5 years in three different accounts:

A. 8%/a compounded quarterly

B. 6%/a compounded monthly

C. 4%/a compounded annually

a) Create a table of values for each account. Calculate four values, representing the value of the investment at the end of four different months. Use these values to create the equation of the exponential regression function that models the investment.

b) Is there a connection between the base of the exponential regression function and the interest rate for each investment? Explain.

c) What will be the future value of each investment after 5 years?

3. Charmaine has just graduated from college. The interest-free period of her student loan has come to an end. Charmaine must now pay $500 per month at an interest rate of 6%/a compounded monthly.

a) The loan manager gave Charmaine the following equation so she could determine how long it would take to pay off her loan

$$(1.005)^{-n} = 0.65$$

where n represents the number of months. Determine the number of months it will take Charmaine to pay off her loan.

b) What is the total amount that Charmaine will end up paying the bank? Explain how you determined your answer.

c) What would you need to know if you wanted to determine the total amount of interest that Charmaine will end up paying on her loan

PRACTISING

4. A pair of shoes costs $150 today. Assuming a constant inflation rate of 4%/a, how much will a similar pair of shoes cost in each of the next 5 years?

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 395 posted in 11th pass on 12.07 12/7/11 2:03 PM

End of Year	Amount Owed ($)
1	2157.84
2	2330.47
3	2516.90
4	2718.26

Year since Purchase	Value ($)
0	8
1	13
2	22
3	40
4	72
5	124

Time (years)	Amount ($)
0	1500.00
1	1560.00
2	1622.40
3	1687.30
4	1754.79

5. Gerry took out a loan to buy a used dirt bike. He does not have to make any payments until the 5th year. The table at the left shows how much he will owe at the end of each of the next 4 years.

a) What is the annual interest rate that Gerry is paying on his loan? Explain how you determined your answer.

b) How much did the dirt bike cost? Explain how you know.

6. Kevin purchased his favourite hockey card for $8 in 2005. He has been tracking the value of the hockey card every year since he bought it.

a) Determine the equation of the exponential regression function that models this situation.

b) Use interpolation to determine the approximate value of the card:

i) 6 months after purchase **iii)** 30 months after purchase

ii) 18 months after purchase **iv)** 42 months after purchase

7. Paula put $1500 into a bank account that pays compound interest annually. Her bank gave her yearly figures for her investment over the next few years, as shown in the table at the left.

a) Explain how Paula can determine an exponential growth function that models the growth of her investment.

b) What is the annual interest rate that is being paid on the account?

c) Assuming the same growth rate, how much money will Paula have in her account after 10 years?

8. Carl invested $10 000 in a guaranteed investment certificate that pays 5%/a compounded annually over the next 4 years.

a) How much will Carl's investment be worth at the end of each of the next 4 years?

b) Write the equation of a function that can be used to model the growth of Carl's investment over x years.

c) Use your equation from part b) to determine the value of Carl's investment after 8 years, assuming that the interest rate remains the same over the entire time.

9. Explain how the compounding period affects the value of i in the following compound interest equation:

$$A(n) = P(1 + i)^n$$

Provide an example that supports your explanation.

10. For each investment, define an exponential growth function in the form

$$A(n) = P(1 + i)^n$$

and state the units for the compounding period.

a) $1000 invested at 6%/a compounded semi-annually

b) $1500 invested at 5%/a compounded monthly

c) $4200 invested at 4.5%/a compounded weekly

d) $2500 invested at 2.5%/a compounded daily

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 396 posted in 11th pass on 12.07 12/7/11 2:03 PM

11. Solve each investment equation for the number of compounding periods, n. Round your answer to the nearest tenth.

a) $120 = 100(1.03)^n$ **c)** $4480 = 1750(1.12)^n$

b) $3200 = 2500(1.046)^n$ **d)** $2950 = 1200(1.074)^n$

12. Merielle recently opened her own restaurant. Her accountant told her that, for tax purposes, the depreciation rate of her restaurant equipment will be 20%, starting from the 2nd year.

a) Suppose that Merielle's restaurant equipment is worth $35 000 at the beginning of the 2nd year. Determine the value of her equipment over the next 6 years.

b) Use your values from part a) to determine the equation of the exponential regression function that models this situation.

c) How much will her equipment be worth 10 years after the purchase date?

13. Sebastien deposited $5000 into an account for his newborn daughter. The account earns 4%/a compounded monthly.

a) Create a table of values that shows the growth of the investment during the first 5 months.

b) Determine the month when the investment will reach

i) $5500 **ii)** $6000

c) Determine the amount of money that will be in the account after

i) 5 years **ii)** 10 years

14. Denise got a cheque for $1249.99 from her employer for profit sharing. She has two options for investing the money:

A. 7%/a compounded annually for 2 years

B. 6%/a compounded monthly for 2 years

Which option should she choose? Explain.

15. Vincenzo needs to borrow money to buy new furniture. He is unsure whether he should borrow from the store or a bank. The time required to repay the loans, based on his monthly payments, can be represented by the following equations, where n is the number of months:

- financing from the furniture store: $0.16 = 1 - (1.004)^{-n}$
- financing from the bank: $(1.0035)^n = \dfrac{1}{0.8425}$

a) Determine the time, to the nearest month, that it would take Vincenzo to repay each loan.

b) What factors might contribute to his choice?

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 397 posted in 11th pass on 12.07 12/7/11 2:03 PM

16. Chao borrowed $170 000 to buy a condo in Calgary, Alberta. The bank told him that his monthly payments will be $1500. The bank agreed to lock the payments at an interest rate of 8%/a, compounded semi-annually, until the debt is repaid. Determine how long, in years and months, it will take Chao to repay the debt. Use the equation

$$(1.006\,558\,197)^{-n} = 0.256\,737\,674$$

where n represents the number of months.

17. Paul has a debt of $110 000. He has decided to consolidate the entire debt into his mortgage. His bank manager has adjusted his monthly mortgage payments to $1000. Determine how long, in months, it will take for Paul to pay off the debt. Use the equation

$$(1.004\,938\,622)^{-n} = 0.456\,751\,580$$

where n represents the number of months.

18. Emma just bought a new car. Her grandfather has always told her that a vehicle is a poor investment because it depreciates the moment that it is driven off the lot. He showed her this equation as a simplified model of how her car might depreciate each year

$$A(t) = 24\,000(0.85)^t$$

where t represents time in years and $A(t)$ represents the value of the car.

a) What did Emma pay for her new car?

b) What is the annual depreciation rate that her grandfather applied to her car?

c) Emma borrowed $20 000 from the bank to buy her car. She agreed to make monthly payments of $300 to pay off the loan. Determine how long, in months, it will take her to pay off the loan. Use the equation

$$(1.009\,166\,667)^{-n} = 0.388\,888\,867$$

where n represents the number of months.

d) How much interest will Emma end up paying on the loan?

e) How much will Emma's car be worth when she pays off the loan, assuming that her car depreciates using her grandfather's model?

19. Indyra is buying a country home in Grande Prairie, Alberta. She has negotiated a mortgage for $150 000, with monthly payments of $1200. She has agreed to an interest rate of 6%/a, compounded semi-annually, until the mortgage is repaid. Determine how long, in years and months, it will take to repay the mortgage. Use the equation

$$(1.004\,938\,622)^{-n} = 0.382\,672\,250$$

where n represents the number of months.

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 398 posted in 11th pass on 12.07 12/7/11 2:03 PM

20. Jason and Jamie have decided to buy a rental property together. They have secured a $185 000 mortgage, with monthly payments of $900.

a) Determine how long, in years and months, it will take them to pay off the mortgage. Use the equation

$$(1.004\ 123\ 915)^n = 6.565\ 713\ 960$$

where n represents the number of months.

b) How much interest will they have paid by the time they have paid off the mortgage?

21. Clorinda has decided to accept her company's buy-out package of $100 000. On her last day of work, she meets with her bank manager to get financial advice. She wants to know if she should use the buy-out money to pay off her mortgage or if she should invest the money in an account that earns 4.5%/a compounded monthly.

a) If Clorinda invests the money, how much will she have in her account after 5 years?

b) Determine how many months it will take Clorinda to pay off the rest of her mortgage if she applies the buy-out money towards her mortgage. Use the equation

$$(1.003\ 305\ 890)^{-n} = 0.669\ 411$$

where n represents the number of months.

Closing

22. a) Explain how you could use an exponential growth function to model the following investment situation.
Principal: unknown
Value of investment after one year: $5000
Interest: 12%/a compounded monthly

b) Once you have a model, what kinds of problems related to the investment situation could you solve? Use examples in your answer.

Extending

23. Your manager has promised you a raise over the next 3 years. You currently earn $40 000 per year. Your manager has given you three different options for how your raise could be paid.

A. 6% in the first year, 4% in the second year, and 2% in the third year

B. 4% in the first year, 4% in the second year, and 4% in the third year

C. 2% in the first year, 8% in the second year, and 2% in the third year

a) Your manager tells you that it does not matter which option you choose, because all three options will give you a 12% pay raise over the 3 years. Do you agree or disagree with your manager? Explain.

b) Which option should you choose? Explain your choice.

Math in Action

The Rule of 72

You can estimate the value of a compound interest investment by using the Rule of 72.

For example, Sarah invests $1000 at 4%/a compounded annually. To estimate how long it will take for her investment to double in value, divide 72 by the annual interest rate: $\frac{72}{4}$ or 18. It will take about 18 years for Sarah's investment to become $2000.

- Suppose that you invested $2000 at an interest rate of 3%/a compounded annually. Estimate the value of your investment after 24 years and 48 years. Use the compound interest formula to check your estimates. $A = P(1 + i)^n$
- What do you notice about the accuracy of your estimates?
- Estimate how long it will take for $2000 to double in value at 3%/a compounded semi-annually, quarterly, and monthly. Check your estimates.
- How is the compounding frequency related to the accuracy of the estimates?

7153_Ceng_M12_C6A_L6.5_pp386-399.indd 399 posted in 11th pass on 12.07 12/7/11 2:03 PM

6 Chapter Self-Test

1. Match each function with the corresponding graph below. Provide your reasoning.

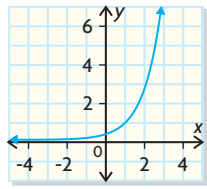
a) $y = 0.2(0.4)^x$

b) $y = 3\left(\frac{1}{3}\right)^x$

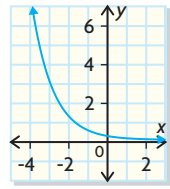
c) $y = 2(4)^x$

d) $y = 0.3(3)^x$

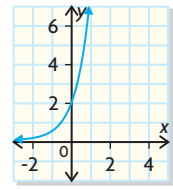
i)



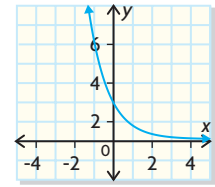
ii)



iii)



iv)



2. Solve each equation, and verify your solution.

a) $5^{(x-2)} = \left(\frac{1}{125}\right)^x$

b) $(0.3)^x = 7^{(x+3)}$

3. At 1:00 p.m., a biologist starts an experiment with 300 yeast cells. Based on similar experiments, she estimates that the number of yeast cells in the culture will double every 30 min.

a) Create a table of values that models the growth of the culture.

b) Use exponential regression to determine a function that models this growth.

c) Estimate how many yeast cells there will be after

i) 4 h **ii)** 12 h

d) How long, to the nearest tenth of an hour, would you expect it to take the culture to grow to more than 100 000 yeast cells?

4. The table at the left shows the Canadian government's net debt, in billions of dollars.

Year	Net Federal Debt ($ billions)
1955	17.56
1960	20.40
1965	26.84
1970	35.82
1975	55.13
1980	110.61
1985	250.52
1990	406.61
1995	550.69
2000	561.73

Statistics Canada

a) Create graphical and algebraic exponential models for the data.

b) What was the approximate net federal debt in 1988, to the nearest hundredth of a billion dollars?

c) Assuming the same growth rate, when did the net federal debt reach $600 billion?

5. Angie borrowed $21 400 to start her own business. Her monthly loan payment is $400. The bank is charging interest on her loan at the rate of 2.4%/a compounded monthly.

a) Determine the number of months it will take Angie to repay the loan. Use the following equation:

$$(1.002)^{-n} = 0.893$$

b) What will be the total amount of interest that Angie will pay on the loan?

***WHAT** DO You Think Now?* Revisit **What Do You Think?** on page 333. How have your answers and explanations changed?

7153_Ceng_M12_C6A_ST_pp400.indd 400 posted in 11th pass on 12.07 12/7/11 4:04 PM

6 Chapter Review

Frequently Asked Questions

Q: What does a half-life function represent?

A: A half-life function is an exponential function that models decay in a real-life situation. In a situation where A_0 represents the initial measurement, t is time since the initial measurement, and h represents the half-life time, the half-life function has this form:

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

For example, carbon-14 takes about 5730 years to decay to half of its initial amount. This means that $h = 5730$ for carbon-14. Suppose that an artifact had 100 g of carbon-14 initially. The mass of carbon-14 in the artifact at time t after the initial date can be determined by using the following equation:

$$A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

Study Aid
- See Lesson 6.3, Example 3.
- Try Chapter Review Questions 12 and 13.

Q: How can you determine the best exponential function of the form $y = a(b)^x$ to model data?

A: Enter the given data into a graphing calculator and do an exponential regression. The exponential regression function will represent the curve of best fit for the data. You can check how well the equation of the regression function fits the data by graphing the function and the points on the same axes.

Study Aid
- See Lesson 6.4, Examples 1 and 2.
- Try Chapter Review Questions 10, 11, and 18.

Q: How do compound interest problems relate to exponential functions?

A: When an investment earns compound interest every month, the interest is calculated based on the value of the investment at the end of each month. So, the original principal and all previous interest earn interest every month. The exponential growth of the investment can be modelled by the function

$$A(n) = P(1 + i)^n$$

where $A(n)$ represents the future value; P represents the principal; i represents the interest rate per compounding period, expressed as a decimal; and n represents the number of compounding periods. Problems that involve compound interest can be solved by using the same methods that are used to solve other exponential functions.

Study Aid
- See Lesson 6.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 14 to 19.

7153_Ceng_M12_C6A_CR_pp401-404.indd 401 posted in 11th pass on 12.07 12/7/11 2:07 PM

PRACTISING

Lesson 6.1

1. **a)** Describe how the end behaviour of exponential functions differs from the end behaviour of polynomial functions.

b) What characteristics do all exponential functions of the form $f(x) = a(b)^x$ have in common?

Lesson 6.2

2. **a)** Predict the characteristics of the graph of the following function:

$$y = 9\left(\frac{1}{3}\right)^x$$

b) What could you change in the equation to make the function an increasing exponential function? Explain.

3. For each exponential function, predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, the range, and whether the function increases or decreases.

a) $y = 125(0.78)^x$ **c)** $y = 3^x$

b) $y = 0.12(0.85)^x$ **d)** $y = 0.85(5)^x$

4. Martha is the head of the safety team at a chemical manufacturing plant. If there is an emergency, Martha informs the five other members of the safety team. Each of these members informs five other employees at the plant, who then inform five more employees, and so on. The way that information about an emergency is passed on can be modelled by the following exponential function:

$$y = 1(5)^x$$

Determine the characteristics of this function.

Lesson 6.3

5. Solve the following equation by writing both sides of the equation with the same base:

$$4(3)^{\frac{3x}{5}} = 108$$

6. Solve each equation, and verify your solution.

a) $3^{2x+7} = 27^{x+2}$

b) $\left(\frac{1}{8}\right)^{3x} = 16^{3-x}$

c) $2^{x-1} = \left(\frac{1}{6}\right)^{x-2}$

7. *E. coli* is a bacterium that lives in the intestines of mammals. In an experiment, the doubling time for a particular strain of *E. coli* was measured as 24 min. The experiment started with about 200 *E. coli* in the sample. The equation that models the growth of this strain of *E. coli* is

$$y = 200(2)^{\frac{x}{24}}$$

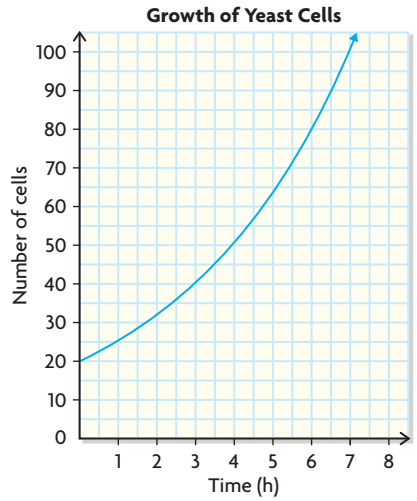
where y represents the number of *E. coli* and x represents the number of minutes since the experiment began.

a) Use the graph of the function to predict how many bacteria there will be in the sample in

i) 12 min **ii)** 12 h

b) Predict how long, in minutes, it takes for the sample to reach 204 800 *E. coli*.

8. In an experiment, the number of yeast cells in a sample was measured each hour. The results are shown in the graph.



7153_Ceng_M12_C6A_CR_pp401-404.indd 402 posted in 11th pass on 12.07 12/7/11 2:07 PM

a) How many yeast cells were in the initial sample?

b) Estimate the number of cells in the sample after 3 h.

c) The equation $y = 20(1.25)^x$ models the growth of the cells in the sample, where x represents time in hours and y represents the number of yeast cells. How many yeast cells will there be in 8 h?

d) Determine in which hour the sample will reach

i) 200 cells **ii)** 1000 cells

9. Many countries have banned the use of the insecticide DDT because of its harmful effects on other wildlife. The mass of DDT that remains at a particular location after 10.00 kg of DDT was sprayed can be modelled by the function

$$f(x) = 10.00(0.944)^x$$

where $f(x)$ represents the mass of DDT in kilograms and x represents the number of years since the spraying.

a) Determine how much DDT, to the nearest hundredth of a kilogram, would be present

i) 5 years after the spraying

ii) 15 years after the spraying

b) When would the mass of DDT reach 5 kg?

c) How long, in years, would it take for less than 1 kg of DDT to be present?

Lesson 6.4

10. On August 16, 1896, gold was discovered near Dawson City, Yukon Territory. The population of Dawson City experienced rapid growth after the discovery. Below is population data from April 1896 to January 1897.

Date	Dawson City Population
April 1, 1896	1 000
July 1, 1896	3 000
October 1, 1896	9 000
January 1, 1897	27 000

a) Determine the equation of the exponential regression function that models the population growth.

b) State the domain and range in this context.

c) Estimate the population of Dawson City in mid-May and in mid-August of 1896. Explain how you determined your answers.

11. A naturalist group has been tracking the deer population near the town of Hudson Bay, Saskatchewan, since 2005.

Years After 2005	Deer Population
1	218
2	237
3	260
4	282
5	307
6	336
7	360

a) Construct a scatter plot for the data.

b) Determine the equation of the exponential regression function that models the data.

c) Assuming the same growth rate, predict the deer population 10 years after 2005.

d) In which year would you expect the population to have doubled?

7153_Ceng_M12_C6A_CR_pp401-404.indd 403 posted in 11th pass on 12.07 12/7/11 2:07 PM

12. Doctors use radioactive tracers when imaging the brain. This technique involves the patient ingesting a radioactive isotope, such as technetium-99. As the isotope moves through the patient's body, doctors can monitor the chemical processes that are taking place in the cells and tissues. Technetium-99 has a half-life of 6.0 h.

a) Write the half-life equation that models the decay of 10 mg of technetium-99.

b) During which hour will the initial 10 mg of technetium-99 have decayed to less than 1 mg?

13. Spears and wooden tools used by early inhabitants were found at an archeological dig site close to Cypress Hills, Alberta. Carbon-14 dating was used to determine the age of the tools. The function that models the decay of carbon-14 is

$$A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

where the initial amount of carbon-14 is 100%, $A(t)$ represents the percent of carbon-14 left in the tools, and t represents the time in years.

a) Based on their experience, the archeologists guessed that the tools were about 8000 years old. If their guess was accurate, what percent of the initial carbon-14 would be present in the tools?

b) The tools were found to have 41% of the original carbon-14 present. Determine the age of the tools to the nearest hundred years.

Lesson 6.5

14. Kaja invested $8000 in an account that pays 6%/a compounded monthly for the next 5 years.

a) Write the function that models the growth of Kaja's investment.

b) Determine how much money there will be in Kaja's account after

i) 30 months **ii)** 48 months

c) In which month will Kaja have $12 000 in her account?

15. Zachary's mother deposited $1000 into an account for him when he was born. She was able to invest this money at 5%/a compounded annually. How much money did Zachary have in this account on his 18th birthday?

16. Explain how changing the compounding period affects how an investment will grow.

17. Principal of $2000 is invested at an interest rate of 9%/a.

a) When will the investment grow to $3200 if the interest is compounded monthly? Express your answer in months.

b) When will the investment grow to $3200 if the interest is compounded daily? Express your answer in days.

18. Yousief has been tracking the selling price of a specific model of truck. His data is shown below.

Estimated Selling Price ($)	19 600	16 700	14 200	12 000	10 200
Age (years)	3	4	5	6	7

a) Determine the equation of the exponential regression function that models the depreciation of the truck.

b) What was the initial selling price of the truck?

c) By what percent does the value of the truck depreciate each year?

19. Darryl makes monthly payments of $1600 on his $250 000 mortgage. The interest rate for his mortgage is 6%/a compounded semi-annually.

a) Determine how many years it will take Darryl to pay off his mortgage. Use this equation:

$$(1.004\,938\,622)^{-n} = 0.228\,340\,313$$

b) What is the total interest that Darryl will have paid by the time he has paid off his mortgage?

7153_Ceng_M12_C6A_CR_pp401-404.indd 404 posted in 11th pass on 12.07 12/7/11 2:07 PM

6 Chapter Task

Caffeine Decay

Many cultures enjoy coffee as part of their daily routine. All coffee, in its original state, contains a substance called caffeine. It takes 4 to 6 h for the human body to reduce the amount of caffeine to half the initial amount ingested. The elimination of caffeine from the human body can be modelled by an exponential decay function.

Every weekday, on her way to school, Maria drinks a triple espresso. Maria is concerned about the amount of caffeine that is present in her body, so she does not drink coffee on Saturday.

? How much caffeine will be present in Maria's body on a Saturday morning?

A. Use research to determine the amount of caffeine in one triple espresso.

B. Determine an exponential function that models the body's ability to reduce the amount of caffeine to half the original amount.

C. Determine an exponential equation that models the amount of caffeine in Maria's body after she drinks one triple espresso.

D. How much caffeine will be present in Maria's body 24 h after her first triple espresso?

E. How much caffeine will be present in Maria's body on a Saturday morning?

Task *Checklist*

- ✔ Did you explain your solution clearly?
- ✔ Did you explain any assumptions you made?
- ✔ Did you use correct mathematical language?
- ✔ Did you reference your sources of information?

7153_Ceng_M12_C6A_CT_pp405-407.indd 405 posted in 11th pass on 12.07 12/7/11 2:06 PM

6 Project Connection

Identifying Controversial Issues

While working on your research project you may uncover some issues on which people disagree. To decide on how to present an issue fairly, consider some questions you can ask yourself or others as you carry out your research.

1. What is the issue about?

Identify which type of controversy you have uncovered. Almost all controversy revolves around one or more of the following:

- Values—What should be? What is best?
- Information—What is the truth? What is a reasonable interpretation?
- Concepts—What does this mean? What are the implications?

2. What positions are being taken on the issue?

Determine what is being said and whether there is reasonable support for the claims being made. You can ask questions of yourself and of others as you research to test the acceptability of values claims:

- Would you like that done to you?
- Is the claim based on a value that is generally shared?

If the controversy involves information, ask questions about the information being used:

- Is there adequate information?
- Are the claims in the information accurate?

If the controversy surrounds concepts, look at the words being used:

- Are those taking various positions on the issue all using the same meanings of terms?

3. What is being assumed?

Faulty assumptions reduce legitimacy. You can ask:

- What are the assumptions behind an argument?
- Is the position based on prejudice or an attitude contrary to universally held human values, such as those set out in the United Nations Declaration of Human Rights?
- Is the person presenting a position or opinion an insider or an outsider?

Insiders may have information and understanding not available to outsiders; however, they may also have special interests. Outsiders may lack the information or depth of understanding available to insiders; however, they may also be more objective.

4. What are the interests of those taking positions?

Try to determine the motivations of those taking positions on the issue. What are their reasons for taking their positions? The degree to which the parties involved are acting in self-interest could affect the legitimacy of their positions.

7153_Ceng_M12_C6A_CT_pp405-407.indd 406 posted in 11th pass on 12.07 12/7/11 2:06 PM

Project Example | Identifying controversial issues

Walther chose pests and invasive species that cause the greatest damage to the natural resources of Western Canada and possible solutions that might be used to control these threats as his topic. Below, he describes how he identified and dealt with a controversial issue.

Walther's Explanation

I found that there is controversy surrounding how the damaged and dying pine tree stands that are infested with the mountain pine beetle are managed. This pest has devastated millions of hectares of pine forests in central British Columbia and western Alberta. The result is a significant amount of standing dead or dying wood, which the logging companies with harvest rights salvage through clear-cut logging. These companies try to get the wood to market before it is no longer useful as lumber, and before the dead dried trees are consumed by wildfires.

There are three controversies regarding salvage logging. These are described in a 2007 report by the Forest Stewardship Council:

1. Environmental interests prefer to see the infested trees left in the woods and allow the natural regeneration process to unfold over time. Economic interests prefer to harvest the infested trees to salvage what they can. Community interests generally want to remove the infested trees to lessen the risk of wildfire.
2. There are concerns that clear-cutting also eliminates green and healthy trees, particularly species other than pine.
3. There are concerns about the practice of "pre-emptive logging," which involves cutting down stands of healthy trees in the hope of removing a route the mountain pine beetles can follow into a new area.

I decided to do more research on the issues above and determine the interests of those taking one side or the other on each issue. I will include a discussion of these issues and offer my opinion on each in my report.

Your Turn

A. Identify the most controversial issue, if any, you have uncovered during your research.

B. Determine the different positions people have on this issue and the supporting arguments they present. If possible, include any supporting data for these different positions.

C. If applicable, include a discussion of this issue in your presentation and report.

7153_Ceng_M12_C6A_CT_pp405-407.indd 407 posted in 11th pass on 12.07 12/7/11 2:06 PM